1 Problem: The Measuring Rod

Three distances are chosen randomly (with uniform probability density) on the interval \((0, 1\) meter). A 1-meter measuring rod is then cut at the three locations corresponding to these three numbers. What is the probability that a quadrilateral can be formed using the lengths of the resulting four segments? (Note for clarity: the quadrilateral must be formed to have a 1 meter perimeter, i.e., the entire length of each segment must be used.)

2 Solution

Let \(X\), \(Y\), and \(Z\) be the locations of each of the cuts on the measuring rod. \(X\), \(Y\), and \(Z\) are independent, identically distributed random variables, each following a uniform distribution on the interval \((0, 1)\). The joint pdf is given by

\[
f_{X,Y,Z}(x, y, z) = \begin{cases} 
1 & x, y, z \in (0, 1) \\
0 & \text{otherwise.}
\end{cases}
\]

If we can specify the region within the unit cube where the values \(x, y, z\) form a quadrilateral and integrate \(f\) over that region, we have our answer. I choose alternatively to define the region in the unit cube where the coordinates \(x, y, z\) will not form a quadrilateral, integrate over this region, and then subtract the result from 1.

Note there are six possible orderings of \(x, y, z\). By symmetry, each ordering has an equal likelihood of \(\frac{1}{6}\). Let event \(A_1\) denote the ordering \(0 < X < Y < Z < 1\). We can form a quadrilateral as long as no one side is longer than the sum of the other three. This only occurs when one of the lengths is longer than \(\frac{1}{4}\) meter. Let event \(N\) be the event we cannot form a quadrilateral. We can denote the probability that we cannot form a quadrilateral and that \(0 < X < Y < Z < 1\) as the union of disjoint intersections:

\[
\Pr\{N \cap A_1\} = \Pr\{(X > \frac{1}{2}) \cup (Y - X > \frac{1}{2}) \cup (Z - Y > \frac{1}{2}) \cup (1 - Z > \frac{1}{2}) \}
\cap \{0 < X < Y < Z < 1\}
\]

\[
= \Pr\{(X > \frac{1}{2}) \cap A\} + \Pr\{(Y - X > \frac{1}{2}) \cap A\} + \Pr\{(Z - Y > \frac{1}{2}) \cap A\}
+ \Pr\{(1 - Z > \frac{1}{2}) \cap A\}
\]

\[
= \int_{\frac{1}{2}}^{1} \int_{x}^{1} \int_{y}^{1} f_{X,Y,Z}(x, y, z) \, dz \, dy \, dx
+ \int_{0}^{\frac{1}{2}} \int_{x+\frac{1}{2}}^{1} \int_{y}^{1} f_{X,Y,Z}(x, y, z) \, dz \, dy \, dx
+ \int_{0}^{\frac{1}{2}} \int_{x}^{\frac{1}{2}} \int_{y+\frac{1}{2}}^{1} f_{X,Y,Z}(x, y, z) \, dz \, dy \, dx
+ \int_{0}^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \int_{y}^{1} f_{X,Y,Z}(x, y, z) \, dz \, dy \, dx
\]

\[
= \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48}
= \frac{1}{12}
\]
We have found the probability we cannot make a quadrilateral \( \text{and} \ x, y, z \) follow the ordering specified by event \( A_1 \). Let \( A_2 \ldots A_6 \) be events corresponding to the other five possible orderings of \( x, y, z \). Because these six events \( (A_1 \ldots A_6) \) form a partition of the sample space, we can employ the law of total probability,

\[
Pr\{N\} = \sum_{i=1}^{6} P\{N \cap A_i\}
\]

\[
= 6 \times \frac{1}{12}
\]

\[
= \frac{1}{2}.
\]

Here, we employ symmetry in the problem to conclude that \( P\{N \cap A_i\} = \frac{1}{12} \) for any ordering \( A_i, i = 1 \ldots 6 \). Therefore, the probability we cannot form a quadrilateral is \( \frac{1}{2} \). It follows that the probability we \text{can} form a quadrilateral is also \( \frac{1}{2} \).