Problem

It is common mathematical knowledge that the derivative of $x^2$, with respect to $x$, is $2x$. One can show this pretty easily, both analytically and graphically.

However, suppose we write $x^2$ as the sum of $x$’s, and then take the derivative:

- Let $f(x) = x + x + \ldots + x$ (x times)
- Then $f'(x) = \frac{d}{dx}[x + x + \ldots + x]$ (x times)
- $f'(x) = \frac{d}{dx}[x] + \frac{d}{dx}[x] + \ldots + \frac{d}{dx}[x]$ (x times)
- $f'(x) = 1 + 1 + \ldots + 1$ (x times)
- $f'(x) = x$

This argument appears to show that the derivative of $x^2$, with respect to $x$, is actually $x$.

Find the fault in the argument.

Solution

The proposed method is not a correct application of the definition of a derivative. A derivative is defined as:

$$\lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta}$$

To correctly compute this we must have $x + \delta$ copies of $x + \delta$ on the left side of the numerator, and $x$ copies of $x$ on the right side of the numerator. The suggested method effectively has only $x$ copies of $x + \delta$ on the left side of the numerator which would not be correct.

The incorrect method is computing this:

In[4]:= Limit[(x + \delta) x - xx \delta] // Simplify, \delta \to 0

Out[4]= x

The correct method would compute this:

In[5]:= Limit[(x + \delta) (x + \delta) - xx \delta] // Simplify, \delta \to 0

Out[5]= 2 \times

Only the second method is consistent with the definition of a derivative.