Nontransitive Dice

Jordan Cuevas

Denote $X_i$ as the number rolled on die $i$. Assuming that the largest number that can appear on a 6–sided die is 6, and that dice are rolled independently of one another, we can rewrite the probability of the first die beating the second die as follows:

$$P(X_1 > X_2) = \sum_{j=1}^{6} P \left( X_1 > X_2 \cap X_2 = j \right)$$

$$= \sum_{j=1}^{5} P \left( X_1 > X_2 \cap X_2 = j \right)$$

$$= \sum_{j=1}^{5} P(X_1 > X_2 | X_2 = j) \cdot P(X_2 = j)$$

$$= \sum_{j=1}^{5} P(X_1 \geq j + 1) \cdot P(X_2 = j)$$

This problem can therefore be restated as selecting 3 dice such that the following system of inequalities is satisfied:

$$\sum_{j=1}^{5} P(X_1 \geq j + 1) \cdot P(X_2 = j) > 0.5$$

$$\sum_{j=1}^{5} P(X_2 \geq j + 1) \cdot P(X_3 = j) > 0.5$$

$$\sum_{j=1}^{5} P(X_3 \geq j + 1) \cdot P(X_1 = j) > 0.5$$

subject to the constraint:

$$\left[ \sum_{j=1}^{6} P(X_1 = j) \right] = \left[ \sum_{j=1}^{6} P(X_2 = j) \right] = \left[ \sum_{j=1}^{6} P(X_3 = j) \right] = 1. \quad (1)$$

There are 462 ways to create a 6–sided die that satisfies Constraint (1). Using a computer loop to search through all possible combinations of these 462 dice chosen 3 at a time, I found 581 unique combinations of 3 dice that satisfy the system of inequalities. One example, chosen at random, is below:

Die 1: \{4, 4, 4, 4, 4, 5\}

Die 2: \{2, 3, 3, 4, 6, 6\}

Die 3: \{1, 1, 5, 5, 5, 5\}

$$P(X_1 > X_2) = \frac{19}{36} > 0.5$$

$$P(X_2 > X_3) = \frac{5}{9} > 0.5$$

$$P(X_3 > X_1) = \frac{5}{9} > 0.5$$

It is worth noting that none of the 581 combinations contain a fair die (each of the numbers 1–6 each appearing exactly once).