Problem of the Week
Week # 28

If $A$ and $B$ are Hermitian matrices such that $e^{A+B} = e^A e^B$, does it follow that $A$ and $B$ commute?

NOTE: If $A$ and $B$ commute, then $e^{A+B}$ must be equal to $e^A e^B$, but if $A$ and $B$ do not commute, then $e^{A+B}$ may or may not be equal to $e^A e^B$. What if $A$ and $B$ are “good” linear transformations—is the exponential law equivalent to commutativity in that case? One way to interpret “good” is as Hermitian.

Show that it does not follow by means of a counter-example.

Consider Hermitian matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

We have $e^{A+B} = \begin{pmatrix} e^1 & e^1 \\ e^1 & e^1 \end{pmatrix} = e^A e^B$,

but $AB = \begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix} \neq BA = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

So, $A$ and $B$ do not necessarily commute.