There is a curve $C$ in $\mathbb{R}^2$ that passes through the point $(3, 2)$ and has an additional property: if $L(x_0, y_0)$ is the portion of the line tangent to $C$ at $(x_0, y_0)$ which lies entirely in the first quadrant, then each point $(x_0, y_0)$ of $C$ is the midpoint of $L(x_0, y_0)$. Find $C$.

**Solution:** We represent the curve $C$ as the pair $(x, f(x))$ for an unknown function $f$. The equation of the line tangent to $f$ at $(x_0, y_0)$ is then $y - y_0 = f'(x_0)(x - x_0)$. It has $x$-intercept given by

$$P_x = \left( x_0 - \frac{y_0}{f'(x_0)}, 0 \right),$$

which can be obtained by just setting $y = 0$ and solving for $x$. A similar calculation shows that the $y$-intercept of this line is located at $P_y = (0, y_0 - x_0f'(x_0))$. The line segment $L(x_0, y_0)$ referred to above has endpoints $P_x$ and $P_y$. We need $x_0$ to be the midpoint of this line segment, which we obtain by restricting the above tangent line to the first quadrant. In other words, we need $(x_0, y_0) = \left( \frac{1}{2}P_x, \frac{1}{2}P_y \right)$. The two equations specified here are actually degenerate, so we consider only the first one, which tells us

$$x_0 = \frac{1}{2} \left( x_0 - \frac{y_0}{f'(x_0)} \right) \implies \frac{y_0}{f'(x_0)} = -x_0.$$
This leads us to the separable ordinary differential equation $y = -xy'$ for $y = f(x)$. We get

$$\frac{dy}{dx} = -\frac{y}{x} \implies \frac{dx}{x} = -\frac{dy}{y} \implies -\log y = \log x + C \implies \frac{1}{y} = Cx$$

for some constant $C$. This constant is fixed by the boundary condition $f(3) = 2 \implies \frac{1}{2} = 3C \implies C = 1/6$. Therefore the curve is given by $y = f(x) = 6/x$. 