POTW #15-05 Graph Components

Counting The Number Of Connected Components In A Graph

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Problem

Let $G$ be the graph with vertex set $\mathbb{Z}_n$ in which vertices $u$, $v$ are adjacent (i.e., joined by an edge) if and only if $u$ and $v$ differ by 6. For each $n \geq 1$, determine the number of components of $G$.

Solution

Summary Of Results

The number of components for any value of $n$ is $\min(n, 6)$, this being the minimum of $n$ and 6. Thus we see that for all $n \geq 6$ the number of components is 6.

Analysis

To find the number of components for any value of $n$ we first draw an $n$ vertex complete graph. Such a graph has an edge connecting every pair of vertices. We then delete all the edges which do not conform to the rule $|u - v| = 6$. In the next cell we visualize the resulting graphs for $n = 1, 2, ..., 30$. 
\textbf{In[1]=} \hspace{1em} \texttt{Grid[Partition[ParallelTable[EdgeDelete[
  CompleteGraph[n, VertexLabels -> "Name", EdgeStyle -> {{Red, Thick}}, PlotLabel ->
  Style[Row[\{"n\". n\}], 16, Bold, Italic, FontFamily \"Helvextia\"]],
  DeleteCases[Flatten[Table[s \rightarrow t, \{s, 1, n - 1\}, \{t, s + 1, n\}]],
  x_ \rightarrow y_ \mid y_ - x = 6]], \{n, 1, 30\}, 3], Frame \rightarrow \text{All} \} // \text{Panel}]

\begin{figure}
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{n=1} & \textbf{n=2} & \textbf{n=3} \\
\hline
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} \\
\hline
\textbf{n=4} & \textbf{n=5} & \textbf{n=6} \\
\hline
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} \\
\hline
\textbf{n=7} & \textbf{n=8} & \textbf{n=9} \\
\hline
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} \\
\hline
\textbf{n=10} & \textbf{n=11} & \textbf{n=12} \\
\hline
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0,0) {1};
\end{tikzpicture} \\
\hline
\end{tabular}
\end{figure}
From this visualization we see that the number of connected components is min(n, 6), this being the minimum of n and 6. We check this result in the next cell and see that it is true.

\[
\text{ParallelTable}[
    \text{Length[ConnectedComponents[EdgeDelete[CompleteGraph[n], DeleteCases[Flatten[
        Table[s -> t, \{s, 1, n-1\}, \{t, s+1, n\}\}, x_ -> y_ /; y - x = 6\}\}, \{n, 1, 100\}]]}
\]