Let $z$ and $w$ be two complex numbers such that $\overline{z}w \neq 1$, where $\overline{z}$ denotes the complex conjugate of $z$. Prove that if $|w| < 1$ and $|z| < 1$, then
\[
\left| \frac{z - w}{1 - \overline{z}w} \right| < 1.
\]

**Hint:** First show why you can assume $z$ is real.

**Solution:** Suppose the result holds for $z$ real; that is, suppose
\[
(1) \quad \left| \frac{r - w}{1 - r\overline{w}} \right| < 1 \quad \text{for } z \in \mathbb{R}, w \in \mathbb{C}, |z|, |w| < 1.
\]
If $z \in \mathbb{C}$, then $z = re^{i\theta}$ for $0 < r < 1$, $\theta \in (0, 2\pi)$. Define $w' = we^{-i\theta}$; then $|w'| = |w| < 1$, so
\[
\left| \frac{z - w}{1 - \overline{z}w} \right| = \left| \frac{r e^{i\theta} - w' e^{-i\theta}}{1 - r e^{-i\theta} w' e^{i\theta}} \right| = \left| e^{i\theta} \frac{r - w'}{1 - r w'} \right| = \left| \frac{r - w'}{1 - r w'} \right| < 1,
\]
by (1). Hence we can proceed under the assumption that $z = r \in \mathbb{R}$. We have
\[
\left| \frac{r - w}{1 - r w} \right| < 1 \iff \frac{|r - w|^2}{|1 - r w|^2} < 1 \iff \frac{(r - w)(r - \overline{w})}{(1 - r w)(1 - r \overline{w})} < 1,
\]
which (cross-multiplying and expanding) is true if and only if $r^2 - rw - r \overline{w} + |w|^2 < 1 - rw - r \overline{w} + r^2|w|^2$; or, more simply, if $r^2(1 - |w|^2) < 1 - |w|^2$. Canceling the factor of $1 - |w|^2$ from both sides – which we can do, without needing to switch the inequality, since $|w|^2 < 1$ – we get $r^2 < 1$, implying $|r| = |z| < 1$. 

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