PO TW  #14-21 Inequality

An Inequality From Number Theory Over The Positive Reals

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Problem

Let $a$, $b$, and $c$ be positive real numbers such that $abc = 1$. Show that $a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$.

Solution

Although most users do not realize it, Mathematica can prove a wide variety of such inequalities. The technique of quantifier elimination, which is implemented in Mathematica’s Resolve function, has been used to prove many theorems of this type, including some which are simply too complicated to be proven by hand.

We want to prove the following statement:

$$\forall_{a,b,c} \text{abc=1} \land a>0 \land b>0 \land c>0 \ a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$$

This can be done as follows using but a single line of Mathematica code.

\begin{verbatim}
In[1]:= Resolve[ForAll[{a, b, c}, a b c == 1 && a > 0 && b > 0 && c > 0, a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3]]
\end{verbatim}

Out[1]= True

This statement is also easy to prove graphically. We replace $c$ by the expression $c = 1/(a \cdot b)$ to reduce the inequality to one involving only two variables as follows:

\begin{verbatim}
In[2]:= a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3 / . c -> \frac{1}{a \cdot b}
\end{verbatim}

Out[2]= \(a^2 + \frac{1}{a^2 b^2} + b^2 \leq a^3 + \frac{1}{a^3 b^3} + b^3\)

Plotting this in the $xy$-plane we see the solid blue color which shows that the inequality is true everywhere.
We note in the next cell that the inequality becomes an equality only in the special case where \( a = b = c = 1 \).