Problem

Imagine a large cube formed by gluing together 27 smaller wooden cubes of uniform size. A termite starts at the center of the face at any one of the outside cubes and bores a path that takes it once through every cube. Its movement is always parallel to a side of the large cube, never diagonal. Is it possible for the termite to bore through each of the 26 outside cubes once and only once and then finish its trip by entering the central cube for the first time? If possible, show how it can be done; if impossible prove it.

Solution

It is not possible for the termite to do this. This is because a three dimensional grid graph having an odd number of vertices does not contain a Hamiltonian cycle. Consider the $3 \times 3 \times 3$ grid graph shown below.
Out[165]= GridGraph[{3, 3, 3}]

\textbf{Mathematica} easily proves in the next cell that this graph contains no Hamiltonian cycles.

\begin{verbatim}
ln[164]= HamiltonianGraphQ[%]
Out[164]= True
\end{verbatim}

An alternative way to prove that the termite cannot accomplish this task is by imagining that the cubes alternate in color like the cells of a three-dimensional checkerboard. The large cube will then consist of 13 cubes of one color and 14 of the other color. The termite’s path is always through cubes that alternate in color along the way; therefore if the path is to include all 27 cubes, it must begin and end with a cube belonging to the set of 14. The central cube, however, belongs to the 13 set; hence the desired path is impossible.

If the termite were allowed to travel \textit{diagonally} then the task could be accomplished. To see this define the following vertices in space.

\begin{verbatim}
ln[166]= v = Flatten[Table[{x, y, z}, {x, 0, 2}, {y, 0, 2}, {z, 0, 2}], 2]
Out[166]= {{0, 0, 0}, {0, 0, 1}, {0, 0, 2}, {0, 1, 0}, {0, 1, 1}, {0, 1, 2},
{0, 2, 0}, {0, 2, 1}, {0, 2, 2}, {1, 0, 0}, {1, 0, 1}, {1, 0, 2}, {1, 1, 0},
{1, 1, 1}, {1, 1, 2}, {1, 2, 0}, {1, 2, 1}, {1, 2, 2}, {2, 0, 0}, {2, 0, 1},
{2, 0, 2}, {2, 1, 0}, {2, 1, 1}, {2, 1, 2}, {2, 2, 0}, {2, 2, 1}, {2, 2, 2}}
\end{verbatim}

Now we can find the shortest traveling salesman tour among these vertices.

\begin{verbatim}
ln[167]= {len, tour} = FindShortestTour[v]
Out[167]= {{27.4142, {1, 10, 19, 22, 25, 26, 27, 24, 23,
20, 21, 12, 15, 18, 17, 16, 7, 4, 13, 14, 11, 3, 6, 9, 8, 5, 2}}
\end{verbatim}

The general shape of the path the termite follows would then appear as shown in the next cell. The termite starts at the red point representing the center of one face of the cube and then bores his way along the blue tunnel until he arrives at the green point at the center of the cube.
Drawing the outlines of the 27 small cubes the path would look like this.
\textbf{Out[151]=} Graphics3D[

\{
  EdgeForm\{\{Brown, Thick\}\}, FaceForm\{Opacity[0]\}, Cuboid/@v, FaceForm\{Opacity[1]\],

  Blue, Thickness[0.02], Tube\{\{\{\{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{2}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}, \{\frac{5}{2}, \frac{1}{2}, \frac{1}{5}\}\}\}, 0.1\},

  Tube\{\{\{\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\}, \{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\}, \{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}\}\}, 0.1\},

  Red, Sphere\{\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\}, 0.1\}], Green, Sphere\{\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\}, 0.1\}],

  ViewPoint -> \{1.01098, -3.22899, 0.0394358\},
  Background -> LightGray, Boxed -> False\} // Panel

\textbf{Out[151]}=