Aircraft Flight Strategies

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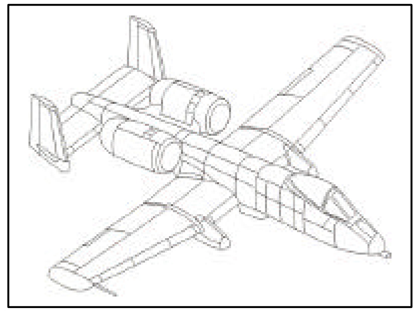
Introduction



Three important considerations in every flight operation are the altitude (possibly variable) at which to travel, the velocity (possibly variable) at which to travel, and the amount of lift that we choose to generate (at the expense of fuel consumption – again possibly variable) during the flight. It turns out that when planning a flight operation, one cannot just choose any desired value for each of these three quantities; they are dependent upon one another. We can relate these three quantities through a set of equations known as the Breguet (pronounced *bre-ga*)¢ Range Equations. These equations are derived in Appendix A. Deriving these equations shows that once we decide to choose constant values for any two of altitude, lift coefficient, and velocity, the third is automatically determined. Thus there are three basic independent flight strategies: constant altitude/constant lift coefficient, constant velocity/constant altitude, and constant velocity/constant lift coefficient. Exercise 1 asks you to analyze how the third quantity must vary under each of these flight strategies.

Commercial flight operations are generally conducted at constant velocity/constant lift coefficient in order to save fuel. In military operations, however, there are often other considerations that override cost efficiency, and thus dictate the choice of a different flight strategy. Surveillance/reconnaissance flights generally dictate flying at constant velocity/constant altitude in order to best gather required intelligence. Phased air operations are sometimes better coordinated when restricted to constant velocity. When several sorties are in the air at the same time, especially both outbound and inbound, safe airspace management often dictates flights at constant specified altitudes. Exercise 2 asks you to more closely analyze which flight strategy may be most appropriate for which military mission. Thus unlike most commercial operations, the military planner must be prepared to operate under any of several different flight strategies.

The following scenarios demonstrate how different techniques of single variable calculus can assist in analyzing the governing equations to yield important information about flight operations. Concepts covered include modeling with

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derivatives, numerical integration, analytic integration, and graphical analysis (of range strategies).

<u>Scenario</u>: **<u>A-10 Close Air Support</u>**

You are the pilot on an A-10 Thunderbolt, Close Air Support (CAS) aircraft. Among the many things for which you are responsible, some of the particular aspects are to determine within what radius your plane can safely service CAS targets, how long it can "loiter" in a target area, and when it must return for refueling.

Now, an interesting aspect of your job is that, at times, some of the instruments malfunction. This forces you to double-check your instruments' accuracy through other means, or to rely on these other means to plan your plane's flight. In this project you are going to answer several questions about the flight of your craft based primarily on your plane's fuel consumption. (Your fuel gauge is known to be working).

Figure 1: The A-10B Thunderbolt

<u>Strategy 1: Flying at Constant Velocity/Constant Lift Coefficient</u>

Range Equation:

You can answer questions regarding how far the plane can travel by relating the distance traveled by the plane to the weight of fuel that it consumes. Assume that you fly at constant velocity and with a constant coefficient of lift (thus, you increase altitude over time as your plane gets progressively lighter). From our knowledge of fluid dynamics, we have the following relationship (this and all following relationships are derived in Appendix A):

$$\frac{dx}{dW} = -\frac{V}{c}\frac{C_L}{C_D}\frac{1}{W},$$

where x = distance traveled, W = weight, V = velocity, c is the coefficient of fuel consumption (c = 0.3700 lbs. of fuel/hr/lb thrust), and the ratio $\frac{C_L}{C_D}$ is 3.839 for constant lift coefficient. Thus, the distance traveled, x, is given by:

$$x = -\frac{V}{c}\frac{C_L}{C_D}\int_{W_{start}}^{W_{finish}} \frac{1}{W} dW .$$

Example 1: You take off weighing 40,434 lbs (this weight includes fuel, armament, and ordnance) and you travel at V = 347.5 mi/hr. You arrive at the target area weighing 36,434 lbs. By use of a numerical integration technique, with an increment size of 1000 lbs in your partition, estimate the distance you have traveled. Does your answer depend on your increment size?

Solution: This requires us to numerically evaluate the integral $-3605.8\int_{40,434}^{36,434} \frac{dW}{W}$,

which we rewrite as $3605.8\int_{35,434}^{40,434} \frac{dW}{W}$. We use the trapezoidal rule

$$x = 3605.8*(.5*f_0*\Delta W + \sum_{i=1}^{n-1} f_i*\Delta W + .5*f_n*\Delta W)$$

to evaluate the integral, with $f(W) = 1/W$ and $\Delta W = 1000$. Substituting for f

yields: $x = 3605.8*(.5*35,434*\Delta W + \sum_{i=1}^{n-1} 1/W_i * \Delta W + .5*40,434*\Delta W)$, where W_i is the value of W in the $i'th$ subinterval (equal to $W_i = 35,434 + (i-1)*(40,434 - 35,434)$). This technique is implemented in the following spreadsheet:

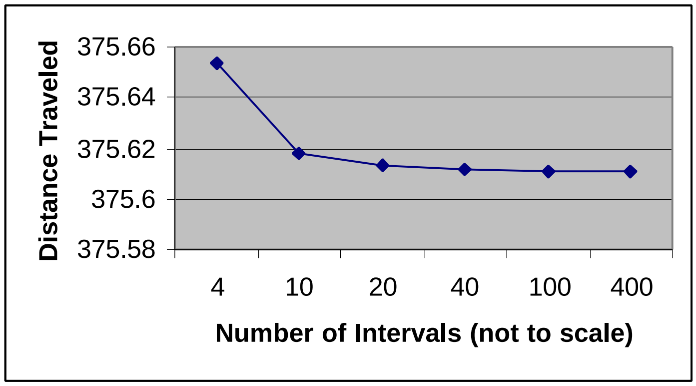
Initial W:	36434			
Final W:	40434			
Intervals:	4			
Delta W:	1000		Distance:	375.6537
W	f(W)	Partial Sum		
36434	2.74469E-05	0.013723		
37434	2.67137E-05	0.040437		
38434	2.60186E-05	0.066456		
39434	2.53588E-05	0.091815		
40434	2.47317E-05	0.10418		

This yields a distance traveled of 375.6 miles. We look to the next example to better answer the question "is the calculated range a function of increment size?"

Example 2: Refine your estimate by increasing the number of partitions. What appears to be the limit as the number of partitions increases without bound?

Solution: Repeating the above process for differing numbers of subintervals yields the following sequence of values for the distance traveled:

Intervals	Distance
4	375.6537
10	375.618
20	375.6129
40	375.6116
100	375.6112
400	375.6112



The calculated range appears to be a monotonically decreasing function of the number of subintervals (or conversely, a monotonically increasing function of increment size). This also appears to be a convergent sequence with a limit of approximately 375.6 miles. Note how few terms are required (in this case) to converge very close to the apparent limit of the numerical integration scheme.

Example 3: Now evaluate the definite integral to find the distance traveled.

Solution: Evaluating the definite integral, which is easy to do for this simple integrand, yields $3605.8\int_{35,434}^{40,434}\frac{dW}{W} = 3605.8 * \ln(W)\,|_{36,434}^{40,434} = 375.6112$ miles. This is in excellent agreement with the numerical solution above.

Endurance Equation

To determine how long you can loiter in the target area with a given amount of fuel, we need to relate the time t to the fuel consumption. With the help of some equations from our fluid dynamics background, we find that, if we assume that we are loitering at a constant velocity, V, and a constant lift coefficient C_L, we have

$$\frac{dt}{dW} = \frac{\frac{dx}{dW}}{\frac{dx}{dt}} = -\frac{1}{c}\frac{C_L}{C_D}\frac{1}{W}$$

Thus, t, the loiter time, is given by:

$$t = \int_{W_{begin}}^{W_{end}} -\frac{1}{c}\frac{C_L}{C_D}\frac{1}{W}dW = -\frac{1}{c}\frac{C_L}{C_D}\int_{W_{begin}}^{W_{end}} \frac{1}{W}dW$$

Example 4*:*
You arrived at the target weighing 36,434 lbs. The S-3 (Air) directs you to reconnoiter the target for 15 minutes (0.2500 hour). How much fuel will you have for your return trip assuming that the plane weighs 29,784 lbs with its armament and ordnance but no fuel?

Solution: Substituting into the endurance equation yields $0.2500 = -10.3757\int_{36,434}^{W_{final}} \frac{dW}{W}$, which we rewrite as $0.2500 = 10.3757\int_{W_{final}}^{36,434} \frac{dW}{W}$.
Evaluating yields $0.2500 = 10.3757(\ln(36,434) - \ln(W_{final}))$. Solving for W_{final} yields $W_{final} = 35,566.6$ lb. This means that we will have 35,566.6 - 29,784 = 5782.6 lbs of fuel remaining when we are ready to return.

Strategy 2: Flying at Constant Velocity/Constant Altitude

For tactical reasons, you are required to return home at constant velocity and constant altitude. You must, therefore, decrease your lift as your plane lightens by decreasing your lift coefficient. It turns out, after some work, that we can derive the relationship

$$\frac{dx}{dW} = -\frac{V}{c\bar{q}SC_{D_o}}\frac{1}{\left(1 + aW^2\right)},$$

where $a = 2.330 \times 10^{-11}$, $C_{D_o} = 0.03700$, $\bar{q}$ = 541.894, S = 506.0 ft^2 (the surface area of the wing), and c = 0.3700 lbs of fuel/hr/lb thrust. Thus, the distance traveled, in miles, is given by:

$$x = \int_{W_{depart}}^{W_{arrive}} -\frac{V}{c\bar{q}SC_{D_o}}\frac{1}{\left(1 + aW^2\right)}dW = -\frac{V}{c\bar{q}SC_{D_o}}\int_{W_{depart}}^{W_{arrive}} \frac{1}{\left(1 + aW^2\right)}dW$$

Example 5:
You have expended all your ordnance, your mission is complete, and you find yourself 478.0 miles away from the airfield. You will return to the field at a constant velocity, V = 460.4 mi/hr, and at a constant altitude. Can you make it home on 4500 lbs of fuel? If so, then how much fuel do you have remaining when you do arrive? If not, then how much additional fuel would you need? Your craft weighs 24,959 lbs when empty of both fuel and ordnance.

Solution: Substituting into the constant velocity/constant altitude equation yields $x = -.12265\int_{29,459}^{24,959}\frac{dW}{(1+2.330\times10^{-11}W^2)}$. Note that we have the freedom here to choose any integration technique (numerical, analytic, Computer Algebra System (CAS)) that we desire. A little experimentation shows that this integral is not going to yield to any of the analytic integration techniques that we (at least most of us) have studied so far. We turn next to our favorite Computer Algebra System (MathCad, Derive, etc…), and find symbolically that $x = -.12265(2.330\times10^{-11})^{-1/2}\tan^{-1}(\sqrt{2.330\times10^{-11}}W)\big|_{29,459}^{24,959}$, or evaluating numerically that $x = 542.546$ miles. Therefore we will make it home with 542.5 - 478.0 = 64.5 miles to spare.

Strategy 3: Flying at Constant Altitude/Constant Lift Coefficient

We have discussed two flight strategies, namely flight at constant velocity/constant lift coefficient, and flight at constant velocity/constant altitude. A third strategy is constant altitude/constant lift coefficient. Now, constant lift coefficient will require you to slow down over time as your plane lightens (otherwise your plane will climb). It turns out for this strategy that we can derive the relationship

$$\frac{dx}{dW} = -\frac{1}{c}\sqrt{\frac{2}{r\,S}}\,\frac{\sqrt{C_L}}{C_D}\,\frac{1}{W^{1/2}}\,dW\,.$$

So the distance that you can travel, in miles, is given by

$$x = \int_{W_{start}}^{W_{end}} -\frac{1}{c}\sqrt{\frac{2}{r\,S}}\,\frac{\sqrt{C_L}}{C_D}\,\frac{1}{W^{\frac{1}{2}}}\,dW = -\frac{1}{c}\sqrt{\frac{2}{r\,S}}\,\frac{\sqrt{C_L}}{C_D}\int_{W_{start}}^{W_{end}}\frac{1}{W^{\frac{1}{2}}}\,dW\,.$$

where $r = 0.002377$ slug/ft^3 (air density) and $\frac{\sqrt{C_L}}{C_D} = 9.997$.

Exercise 4 asks you to compare this strategy to the two already presented. (Hint: You may find that a graphical approach yields the most satisfactory comparison when trying to answer “Does it ever happen that …” type questions.)

Exercises

1. Use the Breguet range equations in Appendix A to determine the following. In each case, explain why your answer is intuitively plausible.
    a. For a constant altitude/constant lift coefficient flight operation, how must the velocity of the aircraft vary during the flight?
    b. For a constant velocity/constant altitude flight operation, how must the lift coefficient of the aircraft vary during the flight?
    c. For a constant velocity/constant lift coefficient flight operation, how must the altitude of the aircraft vary during the flight?

2. For each mission, decide which flight strategy may be best. Explain your reasoning.
    a. Mission: A surveillance/reconnaissance flight conducted at night, designed to gather intelligence about a point target.
    b. Mission: A routine transportation flight, charged with delivering troops and equipment to a designated training area.
    c. Mission: A high priority intercept mission to head off unidentified incoming aircraft and maintain maximum standoff from an aircraft battle group in a hostile theater.
    d. Mission: Routine flight operations in the vicinity of a very busy CONUS airfield.

3. If we have only a limited amount of fuel on board, which of the three flight strategies allows you to travel the furthest? Is any one of the three always best? Is any one of the three always the worst?

4. The Voyager was the first aircraft successfully flown non-stop around the world. How do you think the Breguet equations (along with other design considerations) played a role in the design of this unique aircraft for this very specialized mission?

5. Repeat Requirements 1 through 3 for the F-15E Eagle, using the aircraft data found in Appendix B.

References

[1] Anderson, John D., *Introduction to Flight*, 3d Ed., New York, McGraw-Hill, 1989.

[2] Millikan, Clark, B., *Aerodynamics of the Airplane*, New York, Wiley, 1941.

[3] Perkins, Courtland D., and Robert E. Zhage, *Airplane Performance, Stability, and Control*, New York, Wiley, 1949.

APPENDIX A: DERIVATION OF THE BREGUET RANGE AND ENDURANCE EQUATIONS

1. Mathematical Model:

Lift (L) = Weight of the aircraft (W) (by Newton's second law, assuming no or negligible vertical acceleration)

Thrust (T) = Drag on the aircraft (D) (by Newton's second law, assuming no or negligible horizontal acceleration)

Velocity (V) = dx/dt (where x is the position of the plane at time t)

-dW/dt = cT (loss of weight, all due to fuel consumption, is directly proportional to the thrust produced; c is the specific fuel consumption in units of lbs fuel/(hr x lbs thrust))

2. Definitions:

Coefficient of lift: $C_L = \frac{L}{\bar{q}S}$

Coefficient of drag: $C_D = \frac{D}{\bar{q}S}$

$C_D = C_{D_0} + KC_L{}^2$, where $\bar{q} = \frac{1}{2}rV^2$, $r =$ air density, S =wing area, and C_{D_0} and K are constants.

3. Derived Relationships:

$$\frac{L}{D} = \frac{C_L}{C_D}$$

$$T = D = W\frac{D}{L} = W\frac{C_D}{C_L}$$

$$V = \sqrt{\frac{2W}{rSC_L}}$$

4. Range Equation for Constant Altitude (r constant) and constant C_L:

$-\frac{dW}{dt} = -\frac{dW}{\frac{dx}{V}} = cT$, or $-\frac{dW}{dx} = \frac{cT}{V}$ and $\frac{dx}{dW} = -\frac{V}{cT}$.

By substituting for V: $\frac{dx}{dW} = -\frac{1}{c}\sqrt{\frac{2W}{rSC_L}}\frac{C_L}{C_D}\frac{1}{W} = -\sqrt{\frac{2}{rSC_L}}\frac{C_L^{1/2}}{C_D}\frac{1}{W^{1/2}}$

5. **Range Equation for Constant Velocity and Constant C_L:**

$$\frac{dx}{dW} = -\frac{V}{c}\frac{C_L}{C_D}\frac{1}{W}$$

6. **Range Equation for Constant Velocity and Constant Altitude:**

$$\frac{dx}{dW} = -\frac{V}{cT} = -\frac{V}{cD}$$

Substituting for Drag, where $D = \bar{q}SC_D = \bar{q}S(C_{D_0} + KC_L^2)$ and $C_L = \frac{W}{\bar{q}S}$ yields:

$$\frac{dx}{dW} = -\frac{V}{c(\bar{q}SC_{D_0} + \frac{KW^2}{\bar{q}S})} = -\frac{V}{c\bar{q}SC_{D_0}}\frac{1}{(1+aW^2)}$$, where $a = \frac{K}{\bar{q}^2S^2C_{D_0}}$.

7. **Endurance Equation for a Jet Aircraft at Constant C_L:**

$$\frac{dt}{dW} = -\frac{1}{cT} = -\frac{1}{c}\frac{C_L}{C_D}\frac{1}{W}.$$

APPENDIX B: AIRCRAFT DATA FOR THE F-15E EAGLE

Fuel consumption (lb/hr/lb)	0.9
C_L/C_D	6.193
Take Off Weight (lb)	62,323
Arrival Weight (lb)	58,323
Flight Velocity (mi/hr)	347.5
Aircraft Weight (no fuel, with ordnance)	49,200
a (1/lb^2)	5.866E-11
C_{D_0}	0.026
$\bar{q}$ (lb^2/ft^2)	518.503
S (ft^2)	608
Aircraft Empty Weight (lb)	31,700
Flight Velocity (mi/hr)	347.5
Distance (mi)	325
$\sqrt{C_L}/C_D$	13.928