Tank Munitions and Vector Calculus

Richard Jardine

Introduction

Success in a tank battle is often determined by the superiority of the weaponry involved. The increased lethality of modern tanks is due to the effectiveness of armor piercing projectiles, such as that shown in Figure 1. Army research laboratories continue to improve performance of armor penetrators. At the foundation of the research are mathematical models involving vector calculus.

Figure 1. Tank round travelling in excess of 1500 meters per second, photo courtesy of Army Research Laboratories (ARL)

In this chapter, we describe the use of vector calculus to model armor piercing tank rounds. Along the way, we review mathematical operations with vectors, as familiarity with the mathematics is prerequisite to understanding the application. Mathematical coverage begins with the basic operations of vector arithmetic and continues with exercises applying the dot product, cross product, and gradient. Vector calculus is specifically applied to the modeling of sabot rounds, a type of tank ammunition described in the following paragraphs. In order to discover the connection between the mathematics and the application, it is important that you do the Exercise Sets. You may have to refer to your multivariable calculus text to do the exercises. For this reason, keeping your textbooks for future references is probably a good idea.

To penetrate the massive armor plating protecting the crew of modern tanks, a projectile must be fast-moving, aerodynamic, and have massive momentum. The sabot round pictured in Figure 2 is such a projectile. Do not let the slim profile mislead. The projectile is made of extremely dense materials and propelled to achieve incredible speeds, giving the penetrator the momentum necessary to puncture protective armor tens of centimeters thick.

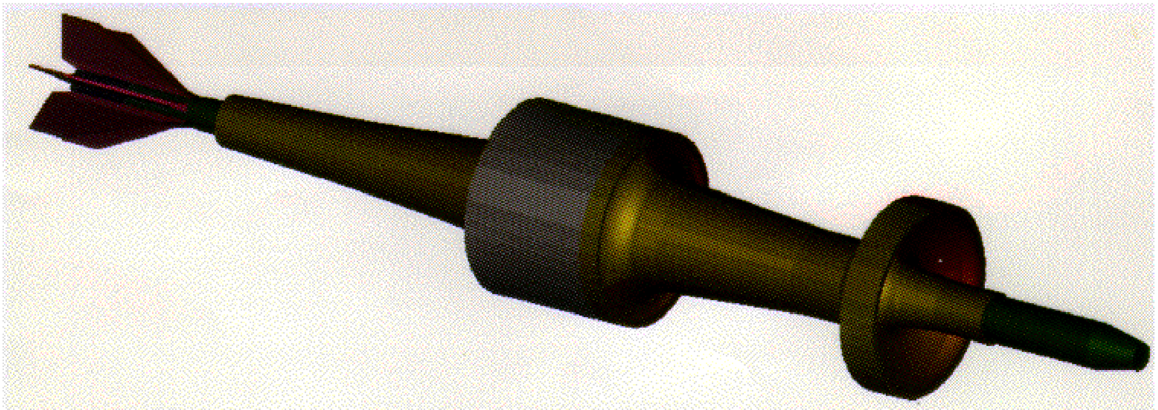


Figure 2. Computer model of penetrator with sabot, courtesy of ARL.

The slender penetrator must be guided down the greater diameter tank main gun barrel.   The guiding sleeve, or sabot, consists of petals which break apart upon exit from the gun tube, as in Figure 3. Optimal design of the sabot round must ensure the sleeve separates from the penetrator without interfering with the flight dynamics of the penetrator, which will speed to a target. Testing the design by repeated firing of the very expensive rounds is cost ineffective. Computer simulations are an efficient alternative to the destructive and costly trial-and-error methods. Programmed into the simulations are algorithms implementing applications of the vector calculus.

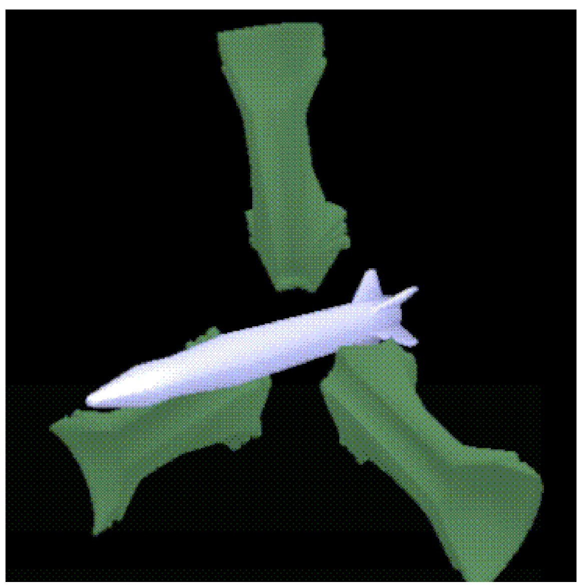


Figure 3. Model of sabot petal separation, courtesy of ARL

The computer models are quite sophisticated and well-developed. Finite element (meshing) methods are used to discretize and model the sabot round, as shown in Figure 4. Each of the apparent rectangles are in fact quadrilaterals. Each quadrilateral is the external face of a finite element brick. The bricks are the building blocks of the model of the sabot round. Depending on the details required in the simulations, the sabot round model can be divided into thousands of these bricks (meshes) by computer.

The Model

In the milliseconds after the explosion which propels the sabot round out the tube of the tank cannon, the round is subject to intense heat and pressure. In order to model the influence of the heat and fluid pressure on the flight dynamics of the projectile, millions of computations are performed in the process of the computer simulation. The effects of heat and pressure on the blocks are simulated using vectors.

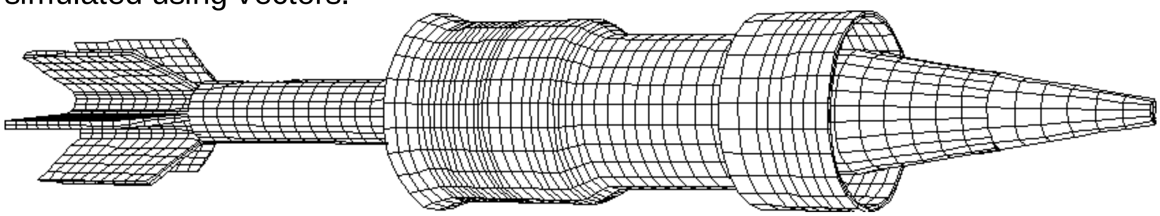


Figure 4. Discretized model of sabot round, graphic courtesy of ARL.

The pressure field affecting the sabot round is modeled with a vector-valued function. As an example, the function

$$\boldsymbol{p}(x,y,z) = p_1(x,y,z)\,\boldsymbol{i} + p_2(x,y,z)\,\boldsymbol{j} + p_1(x,y,z)\,\boldsymbol{k}$$

represents the pressure field that exists in the gun tube. The component of the pressure in the x, y and z directions are p_1, p_2 and p_3, respectively. To be specific with an example, suppose $\boldsymbol{p}(x,y,z) = 2xy\,\boldsymbol{i} + xz\,\boldsymbol{j} + 3y\,\boldsymbol{k}$ is the pressure field function. Then the pressure in the tube at the point $A(1,2,3)$ would be $\boldsymbol{p}(x,y,z) = 4\,\boldsymbol{i} + 3\,\boldsymbol{j} + 5\,\boldsymbol{k}$. Obviously, this pressure field has an influence on the motion of the projectile. With mathematical models of the pressure influence, engineers can determine the flight characteristics of the penetrator.

One mathematical tool used to model the influence of the pressure field on the sabot round is to apply the dot product operation. By calculating the dot product of the pressure field with a unit vector normal to the surface of the projectile, we can pinpoint the pressure at that point on the sabot round. Figure 5 is a vector representation of the dot product computation. Prior to doing that calculation, we review the process of finding a unit vector normal to a surface and the process of evaluating the dot product.

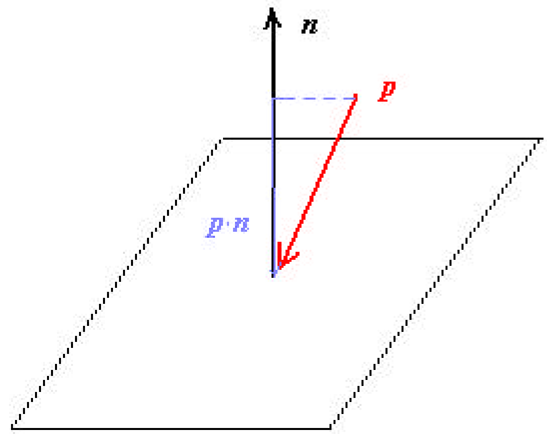


Figure 5. Dot product of ***p*** and ***n*** is the influence of the vector p on the planar surface.

Let's begin by working toward finding the vector normal to the surface of the projectile. Before we do the three-dimensional problem, we start with the simplest two-dimensional case: finding the normal vector to a line. One representation of a line l_1 is the equation $ax + by + c = 0$. Recall the slope-intercept form of the equation of the line, $y = \frac{-a}{b}x - \frac{c}{b}$. From this form we see that the slope m_1 of l_1 is $m_1 = \frac{-a}{b}$. A second line, l_2, perpendicular to l_1 would have slope $m_2 = \frac{b}{a}$, the negative reciprocal of m_1.

EXERCISE SET 1: Given the line l_1: 3x + 2y + 4 = 0,

a. Plot the line

b. Find the slope of the line l_1.

c. Find the slope of a line l_2 perpendicular to l_1.

d. Plot the line l_2 on the same graph as l_1.

A vector ***v*** coincident with (or parallel to) the line l_1 has components $\boldsymbol{v} = b\,\boldsymbol{i} - a\,\boldsymbol{j}$. Of course, this is not the only vector parallel to l_1, as any scalar multiple of ***v*** will be parallel to ***v*** and l_1. Unit vectors are used to standardize operations with vectors. A unit vector has length one. For example, the unit vector for $\boldsymbol{v}$ is the vector $\boldsymbol{u} = \frac{b}{\sqrt{a^2+b^2}}\boldsymbol{i} + \frac{-a}{\sqrt{a^2+b^2}}\boldsymbol{j}$.

EXERCISE SET 2: Given the line l_1: $3x+2y+4=0$,

a. Use $\boldsymbol{v} = b\,\boldsymbol{i} - a\,\boldsymbol{j}$ to find a vector parallel to l_1 and sketch that vector on l_1.

b. Find a unit vector $\boldsymbol{u}$ parallel to l_1.

c. Find a unit vector $\boldsymbol{n}$ normal (perpendicular) to l_1 and sketch that vector on l_1.

From the last exercise, note that a vector $\boldsymbol{n}$ normal to $\boldsymbol{v}$ has components $\boldsymbol{n} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$. A general way to obtain that result is to use the gradient operator , $\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$. You may recall that role of the vector from the multivariable calculus. By evaluating the gradient of the linear function $f(x,y) = ax+by+c$, we obtain the normal vector to the line l_1. This result can be generalized not only to functions other than linear functions, but to surfaces in three dimensions. In doing the following exercises you will rediscover that application of the gradient.

EXERCISE SET 3:

a. Use the gradient operator to find a vector perpendicular to the line $3x+2y+4=0$.

b. Find a vector normal to the line $7x-2y+6=0$, then sketch the line and the normal vector to the line at the point P(0,3).

c. Plot the parabola $x^2+5x-7y=0$ and the unit normal vector at the point P(2,2).

d. Plot the outward unit normal to the circle of radius 3 at the point $P\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

e. Sketch the plane $3x+2y+5z=10$ and the unit normal to the plane at the point $P(1,1,1)$. Note that just as $\boldsymbol{n} = a\,\boldsymbol{i} + b\,\boldsymbol{j}$ was normal to the line $ax + by + c = 0$, $\boldsymbol{n} = a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}$ is normal to the plane $ax + by + cz + d = 0$.

f. Sketch the cylinder $z^2 + x^2 = 9$ and find the outward unit normal at the point $P\left(\frac{3\sqrt{2}}{2}, 2, \frac{3\sqrt{2}}{2}\right)$.

A significant portion of the sabot can be modeled with a cylinder, such as that in the last exercise. To find the influence of the pressure field on the cylinder, we use the dot product vector operation. By calculating the dot product of the vector pressure field with the normal vector to the surface, we obtain the scalar projection of the vector field on the unit normal vector. That projection represents the scalar pressure at that point on the surface of the cylinder. Knowing the pressure on the surface of the projectile, engineers can make predictions about the flight dynamics of the sabot round.

EXERCISE SET 4:

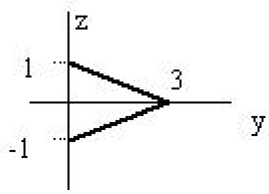
a. Find the dot product of the vectors $\boldsymbol{u} = 3\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{v} = 6\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$.

b. Find a unit vector of $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, then evaluate the dot product of the unit vector with the vector function $\boldsymbol{f}(\mathrm{x,y,z}) = 2\mathrm{xy}\,\boldsymbol{i} + 3\mathrm{z}\,\boldsymbol{j} + 2\mathrm{y}\,\boldsymbol{k}$.

c. A circular ring of diameter 6 exists in a field modeled by the vector function $\boldsymbol{f}(x,y) = 2xy\boldsymbol{i} - 3x\boldsymbol{j}$. Find the effect of the field at the point $P\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

d. The fin of a sabot round is modeled by a portion of the plane $2x - 3z = 0$. Given a pressure field modeled by $\boldsymbol{P}(x,y,z) = \frac{xy}{2}\boldsymbol{i} - 6xy\boldsymbol{j} - 2yz\boldsymbol{k}$, find the pressure at the point $A(3,1,2)$.

e. The nose of the penetrator can be modeled by a cone. Develop the model of a cone with cross section shown in the sketch below. Find the pressure at the point $\boldsymbol{B}\left(\frac{2}{3}, 1, \frac{2}{3}\right)$ on the nose of the penetrator in the field $\boldsymbol{P}(x,y,z) = \frac{xy}{2}\boldsymbol{i} - 6xy\boldsymbol{j} - 2yz\boldsymbol{k}$.



f. A computer program discretizes the penetrator into quadrilateral finite element blocks. The output of the discretization program (Table 1) identifies the coordinates of the four corners of the external face of one block.

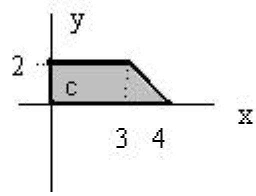
Node	Coordinates
1	(2,1,1)
2	(3,4,1)
3	(1,5,2)
4	(0,1,2)

Table 1

(1) Find the pressure of each corner of the face in the pressure field $\boldsymbol{P}(x,y,z) = \frac{xy}{2}\boldsymbol{i} - 6xy\boldsymbol{j} - 2yz\boldsymbol{k}$. Use the cross product to find a vector normal to the plane passing through 3 of the points, or find the equation of the plane passing through 3 of the points and use the gradient to find the normal. Then evaluate the dot product of the pressure vector with the normal to the plane at each point of interest. How might the pressure on that portion of the projectile (on the entire finite element block) be represented?

(2) Is there a better method to represent the pressure on the element? For example, consider evaluating the pressure at the centroid of the element face.

g. A fin of the penetrator is shown in the sketch below. The temperature affecting the fin is modeled with the function $T(x,y,z) = 3x^2 + 4y^2 + 7z - 9$. The resistance to heat flow determines the distribution of the composite materials that comprise the fin. Sketch the isotherms (curves of constant temperature) on the fin. Then find the direction of maximum temperature change from the point $\boldsymbol{C}(1,1)$. Plot a unit vector in that direction. What portion of the fin should be composed of the most heat resistant materials?



Conclusion

The last exercise and this chapter have demonstrated the applicability of elementary vector calculus operations in understanding modern tank munitions from an engineer's perspective. Engineers at the Weapons Technology Directorate of the Army Research Laboratory design and test prototype projectiles using sophisticated supercomputing facilities. Vector calculus computations are coded into those computer simulations; programs used to design and evaluate present and proposed penetrators. Cadets and faculty from the United States Military Academy have participated in the research effort in conjunction with the ARL scientists. The success of future weapons development is contingent on the ability of civilian and military researchers to effectively use mathematical tools.

References

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