King of Battle: The Cannon

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Introduction

In this section, we discuss some of the basic considerations in firing artillery rounds: the motion of the projectile and the recoil effects on the weapon system. The model development for these two considerations is done through a historical context. In essence, “the cannon problem” has been part of the West Point mathematics curriculum for centuries. These artillery issues were studied at West Point by officer candidates before the Military Academy was established at West Point in 1802. George Baron, a professor of mathematics, taught artillerists the mathematics of projectile motion, the proper placement of the weapon to handle recoil, and other basic mathematics as early as 1800 at West Point.

Historical Setting

On 1 July 1824, a cadet, who many years later would produce a controversial rifled cannon, graduated from the United States Military Academy. Robert Parker Parrott (1804-1877) was commissioned upon graduation as a Second Lieutenant of the Artillery. After a tour of teaching at the Academy and a tour of garrison duty, he changed his branch to Ordnance and served as an inspector of cannon during the period 1834-35. In 1836 he resigned his commission after reaching the rank of Captain.

From 1836 until 1867 he served as superintendent of the West Point Foundry, also called the Cold Spring Foundry, located at Cold Spring, New York on the east side of the Hudson River about one half mile north of West Point. The West Point Foundry became such a leading producer of ordnance for both the Army and the Navy that the initials WPF and RPP, which were etched on the muzzle of Parrott cannons, were well known throughout both services.

Parrott rifles saw extensive use in field and siege as well as aboard ship throughout the Civil War. They were praised. They were cursed. Some of the smaller calibers weren't up to the standard of competing rifles and larger models had an unfortunate tendency to burst. But considered collectively, as a rifled weapons system, the Parrott was hard to beat. It was easy to operate by inexperienced cannoneers. It was an extremely tough weapon. It was inexpensive to manufacture and could be produced quickly and in quantity. The Parrott rifle became so renowned during the Civil War that someone suggested the parrot should replace the eagle as the emblematic bird of state.

In this discussion of artillery, we will first look at projectile motion, then we will analyze the effect of firing on the gun system itself.

Projectile Motion

In order to derive a model for projectile motion, we first assume that the projectile behaves like a particle moving in a vertical plane (2 dimensional) and that the only force acting on the projectile during flight is the force of gravity, which points straight down. With these assumptions, we are ignoring many other effects on the motion of the projectile: the ground moving as the earth turns, the air friction and wind effects on the projectile's motion, and the force of gravity changes as the projectile changes altitude. Later, we can consider some of these effects.

Let's assume that the projectile is launched from the origin at time *t = 0* with an initial velocity vector of $\vec{v}_0$, the first component is the the horizontal or *x* direction with unit vector $\vec{i}$ and the second component is in the vertical or *y* direction with unit vector $\vec{j}$. If q is the angle that $\vec{v}_0$ makes with the horizontal axis, then

$$\vec{v}_0 = |v_0|\cos q \ \vec{i} + |v_0|\sin q \ \vec{j} \ . \qquad (1)$$

From our assumptions, the projectile's initial position is

$$\vec{r}_0 = 0\vec{i} + 0\vec{j} \ . \qquad (2)$$

Since we assume that the only force acting on the projectile during its flight comes from a the downward acceleration of gravity of magnitude *g*, we can use Newton's second law to obtain

$$\frac{d^2\vec{r}}{dt^2} = -g \quad \vec{j} \qquad (3)$$

We find the projectile's velocity or position by solving this second order differential equation (3) with the initial conditions: Equation (2) and $\vec{v}(0) = \vec{v}_0$. The solution can be found through integration. We integrate (3) to obtain

$$\frac{d\vec{r}}{dt} = (-gt)\vec{j} + \vec{v}_0$$

Another integration step produces

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} = 1/2 \ gt^2 \ \vec{j} + \vec{v}_0 t + \vec{r}_0 \qquad (4)$$

We substitute the values of $\vec{v}_0$ and $\vec{r}_0$ from Equations (1) and (2) into (4) to obtain

$$\vec{r} = (|\, v_0 \,|\cos q)t \ \vec{i} + (|\, v_0 \,|\sin q \ t - 1/2 \ gt^2) \ \vec{j} \ , \qquad (5)$$

or by component, $x(t) = (|\, v_0 \,|\cos q)t$ and $y(t) = (|\, v_0 \,|\sin q \ t - 1/2 \ gt^2)$.
If we measure time in seconds and distance in meters, *g* is 9.8 m/sec^2 and if we measure time in seconds and distance in feet, *g* is 32 ft/sec^2.

Example 1: Let's see what happens for a specific example. If a projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec at an angle of elevation of 50 degrees, let's find the projectile's location at *t=20*. We can use Equation (4) with $|v_0| = 500$, $q = 50(p/180)$ radians, *g = 9.8*, to find the projectile's coordinates. The two equations are:

$$x(t) = 500\cos(50p/180)\,t \quad \text{and} \quad y(t) = 500\sin(50p/180)\,t - 4.9t^2 . \qquad (6)$$

Therefore, when *t=20* seconds, the projectile is 5700 meters high (*y* component) and 6428 meters down-range (*x* component).

To find when and where the projectile lands when it is fired over horizontal ground, we set the vertical position $(|v_0|\sin q\ t - 1/2\ gt^2)$ equal to zero from Equation (5) and solve for *t*. These roots are *t=0* or $t = 2v_0 \sin q / g$. Since *t=0* is the time the projectile is fired, the other root, $t = 2v_0 \sin q / g$ must be the time when the projectile strikes the ground. For our example, this value is *t=78.16* seconds. To find the projectile's range *R*, the distance from the origin to the point of impact on horizontal ground, we find the value of *x(t)* when *t=78.16*. For our example, $R = 500\cos(50p/180)\ 78.16 = 25{,}120$ meters. Be careful to note that we are not claiming this is maximum range of this weapon system. For projectiles with no air resistance, the maximum range is obtained for a specified initial velocity by firing the projectile at 45 degrees.

Through further analysis and calculation with Equation (5), we can find the highest point of this projectile. This occurs when its vertical velocity component is zero, that is, when $|v_0|\sin q - gt = 0$. For our example, the value of *t* when the projectile is at its highest is *t=39.08* seconds. At that time the trajectory is *H=7,485* meters high.

Now let's look at the shape of the trajectory of this projectile. If we eliminate *t* and combine the two equations for *x* and *y* in Equation 5, we obtain the equation

$$y = -\left(\frac{g}{2v_0^2\cos^2 q}\right)x^2 + (\tan q)x .$$

We see that this equation, for given values of g, q, and v_0, is a parabola in *x* and *y*. Therefore, the trajectory of this ideal projectile (no air resistance) is parabolic. This ideal situation isn't always accurate enough for predicting trajectories of real projectiles, so let's refine the model to consider air resistance.

Air Resistance

Air resistance on moving projectiles can depend on many variables: the velocity of the projectile, the projectile's surface area, the relative densities of the fluid and the projectile, the projectile's shape and smoothness, and the fluid compressibility. At slow speeds the dominant effect providing resistance to motion is the sliding of the fluid over the projectile's surface (viscosity). At these slow speeds, we assume the number of molecules contacting the surface per unit of time is proportional to the product of the projectile's velocity and its surface area. Therefore, a simple model for the resistive force due to friction is the proportionality $F_r = -kv$, where *k* is the constant of proportionality and *v* the velocity of the projectile. The model assumes factors such as surface area and surface smoothness are constant.

At higher speeds another air-resistance model may be more accurate. As the velocity increases, more collisions of the projectile with the fluid's molecules take place. Eventually the effect of the collisions comes to dominate the "sliding" effect predominant at low speeds. The model often used in this case and observed experimentally is that the resistive force *F* is proportional to the square of the velocity. This model is written as $F_r = -kv^2$, where as before, *k* is the constant of proportionality and *v* the velocity of the projectile. This makes the equation nonlinear and, therefore, more difficult to solve analytically. However, graphical and numerical solution techniques may be used for the nonlinear equation case, and some of these methods will be investigated later on in the text.

Projectile Motion with Viscous Friction

Let's develop a model for a projectile moving at low velocities, where viscosity is the dominant resistance and gravity is still the only other force. All other forces are neglected. Using Newton's second law we have, $\sum F = F_g + F_r = ma.$ Making substitutions for a projectile with position $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, we get $m\frac{d^2\vec{r}}{dt^2} = -mg\,\vec{j} - k\frac{d\vec{r}}{dt}$. We further simplify this equation by writing an equation for both the *x* and *y* components of motion. Our equations are:

$$\begin{aligned} m\frac{d^2y}{dt^2} + k\frac{dy}{dt} &= -mg \\ m\frac{d^2x}{dt^2} + k\frac{dx}{dt} &= 0 \end{aligned} \quad . \qquad (7)$$

These two uncoupled, linear equations can be solved separately using various techniques likes conjecturing and undetermined coefficients. The two general solutions are:

$$\begin{aligned} x(t) &= c_1 + c_2 e^{-\frac{kt}{m}} \\ y(t) &= c_3 + c_4 e^{-\frac{kt}{m}} - \frac{mgt}{k} \end{aligned} \quad . \qquad (8)$$

The constants c_1, c_2, c_3, and c_4 are determined from substitution of the initial conditions: $\vec{v}_0 = |v_0|\cos q\,\vec{i} + |v_0|\sin q\,\vec{j}$ and $\vec{r}_0 = 0\vec{i} + 0\vec{j}$. These calculations produce

$$c_1 = \frac{m\,|\,v_0\,|\cos q}{k},$$
$$c_2 = -\frac{m\,|\,v_0\,|\cos q}{k},$$
$$c_3 = \frac{m}{k}(|\,v_0\,|\sin q + \frac{mg}{k}),$$
$$c_4 = -\frac{m}{k}(|\,v_0\,|\sin q + \frac{mg}{k}).$$

Therefore, the solution is

$$x(t) = \frac{m\,|\,v_0\,|\cos q}{k} - \frac{m\,|\,v_0\,|\cos q}{k} e^{-\frac{kt}{m}}$$
$$y(t) = \frac{m}{k}(|\,v_0\,|\sin q + \frac{mg}{k}) - \frac{m}{k}(|\,v_0\,|\sin q + \frac{mg}{k})\, e^{-\frac{kt}{m}} - \frac{mgt}{k} \quad (9)$$

Example 2: Let's see how adding this air resistance term changes our results from Example 1. Using the same data that $|\,v_0\,| = 500$, $q = 50(p/180)$ radians, and *g = 9.8,* along with the new data that the mass of the projectile is *m=3* slugs and the proportionality constant is *k=0.05.* We substitute these values into Eq (9) to obtain:

$$x(t) = 19284 - 19284e^{-0.0167t}$$
$$y(t) = 58261 - 588t - 58261e^{-0.0167t}. \quad (10)$$

We determine maximum range and maximum projectile height using the same techniques as used in Example 1. We obtain the maximum range as *R=12,888* meters (at *t=66.225* seconds) and the maximum height of *H= 5,284* meters (at *t=30.1* seconds).

These values are significantly different (less) than those found in Example 1. Therefore, we see that his much air resistance does effect the trajectory of the projectile. We plot Eqn (5) and (10) on the same axes to see the effects. Figure 1 shows the effects of adding air resistance.

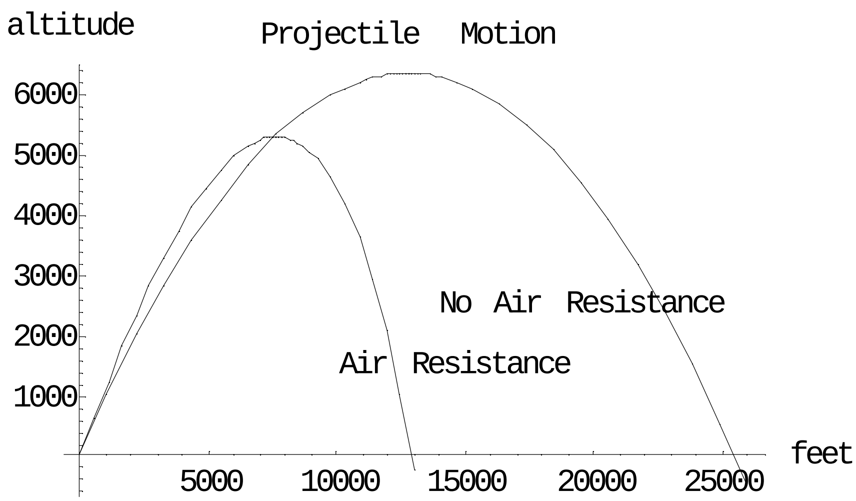


Figure 1: Trajectories of projectile without air resistance (Eqn (5)) and with air resistance (Eqn (10)).

Gun System Model

Let's go back and analyze the Parrott Gun itself. A new experimental 4.2 inch Parrott is to be test fired while mounted on a two-wheel, bracket trail gun carriage. To determine the gas pressure force generated upon firing, the cannon was first test fired at zero degrees elevation from a fixed casemate. With a 30 pound projectile and a 3.25 pound charge, the gas pressure force generated was determined to be 4500 pounds. With this projectile and charge it takes one-fifth of a second for the round to leave the cannon tube (i.e. the gas pressure force lasts for one-fifth of a second).

The cannon and carriage of our Parrott together weigh 4800 pounds. The damping force (that force created by the trails scraping along the ground and retarding the rearward recoil) acts horizontally on the cannon and carriage and is known to be numerically equal to $600\sqrt{2}$ times the instantaneous velocity of the cannon and carriage. The experimental Parrott will be test fired with zero degrees elevation on a range which has a safety wall constructed five feet behind the end of the carriage trails as shown in Figure 2. We produce both a first order and second order model just to compare the results.

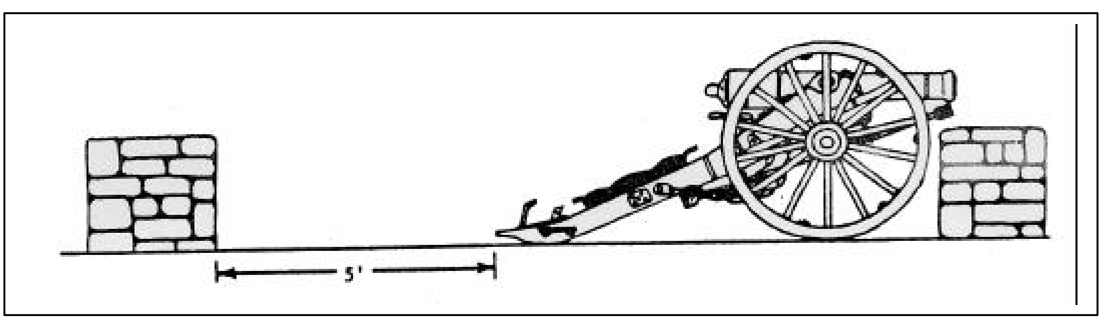


Figure 2: Parrott Gun

First-order Model

Let's analyze this situation. We model the given interaction as a first order differential equation with appropriate initial values. We use velocity of the cannon *v(t)* as the dependent variable and time *t* as the independent variable. We assume the following:

1. The weight of the projectile is negligible in relation to the weight of the cannon.
2. There is no friction in the cannon tube.
3. At $t = 0$ gas pressure in the tube is immediately 4500 lbs.
4. The forces are projected only in the horizontal direction.
5. We use the following notation: F_1 = gas pressure force, F_2 = damping force

We are concerned with the horizontal displacement of the cannon, and there is no horizontal component of the cannon's weight. First, we calculate the mass of the cannon with the formula *mg* = weight; therefore, $m\left[32\frac{ft}{\sec^2}\right] = 4800 lbs$ or *m* = 150 slugs. Then we apply Newton's Second Law, using $a = \frac{dv}{dt}$ for the acceleration of the gun and *v(t)* as the velocity of the gun. We separate the calculations into the two periods of time, $0 \le t < \frac{1}{5}$ and $t \ge \frac{1}{5}$. The following calculations are used to formulate the model differential equation:

$0 \le t < \frac{1}{5}$:

$$\sum F = ma$$

$$F_1 - F_2 = ma$$

$$4500 - 600\sqrt{2}v = 150\frac{dv}{dt}$$

$$30 - 4\sqrt{2}v = \frac{dv}{dt}$$

or

$$\frac{dv}{dt} + 4\sqrt{2}v = 30$$

$t \ge \frac{1}{5}$:

$$\sum F = ma$$

$$-F_2 = ma$$

$$-600\sqrt{2}v = 150\frac{dv}{dt}$$

$$-4\sqrt{2}v = \frac{dv}{dt}$$

or

$$\frac{dv}{dt} + 4\sqrt{2}v = 0$$

Therefore, the differential equation model is:

$$\frac{dv}{dt} + 4\sqrt{2}v = \begin{cases} 30, & 0 < t < \frac{1}{5} \\ 0, & t \geq \frac{1}{5} \end{cases} . \qquad (11)$$

We also need an initial condition for this equation. We note that at *t = 0* the velocity and displacement are both zero. Now we can solve this initial value problem to determine whether or not the carriage trails will strike the safety wall after the cannon is fired.

First-order Solution

The solution of $\frac{dv}{dt} + 4\sqrt{2}v = 30$ on the interval $0 < t < \frac{1}{5}$ is as follows:

This is a separable equation, so we separate variables and integrate both sides of the equation as shown: $\int \frac{dv}{30 - 4\sqrt{2}v} = \int dt$. Letting $u = 30 - 4\sqrt{2}v$, we substitute to get the integral $-\frac{1}{4\sqrt{2}} \int \frac{du}{u} = t + C_3$. The following steps perform the integration and back substitution for this integral:

$$-\frac{1}{4\sqrt{2}} \ln|u| = t + C_3$$

$$-\frac{1}{4\sqrt{2}} \ln\left|30 - 4\sqrt{2}v\right| = t + C_3$$

$$\ln\left|30 - 4\sqrt{2}v\right| = -4\sqrt{2}t + C_2$$

$$30 - 4\sqrt{2}v = C_1 e^{-4\sqrt{2}t}$$

$$v = \frac{30}{4\sqrt{2}} + Ce^{-4\sqrt{2}t}$$

We apply the initial condition to get the particular solution. The equation $v(0) = 0 = \frac{30}{4\sqrt{2}} + C$ is solved for *C* to get $C = -\frac{30}{4\sqrt{2}}$. This is substituted to produce the solution:

$$v(t) = \frac{15}{2\sqrt{2}}\left(1 - e^{-4\sqrt{2}t}\right). \qquad (12)$$

We determine the displacement *x* at $t=\frac{1}{5}$ sec. This involves the integration of the velocity function over the interval: $x\left(\frac{1}{5}\right)=\int_0^{1/5} v(t)dt$. Using Equation (12), we get $x\left(\frac{1}{5}\right)=\int_0^{1/5}\frac{15}{2\sqrt{2}}\left(1-e^{-4\sqrt{2}t}\right)dt$. The integral evaluates to $x\left(\frac{1}{5}\right)=.4256\,ft$.

We can also calculate velocity at $t=\frac{1}{5}$ through substitution to get

$$v\left(\frac{1}{5}\right)=\frac{15}{2\sqrt{2}}\left(1-e^{\frac{-4\sqrt{2}}{5}}\right)\quad =3.5925\frac{ft}{\sec}. \qquad (13)$$

We continue by solving the equation on the second interval of the model in Eqn (11), $t\geq\frac{1}{5}$, using similar steps and Equation (13) as the initial condition: $\frac{dv}{dt}+4\sqrt{2}v=0;\quad v\left(\frac{1}{5}\right)=3.5925$. Separating variables and integrating produces: $\int\frac{dv}{v}=-\int 4\sqrt{2}dt+C_1$ and $\ln v=-4\sqrt{2}t+C_1$. Therefore, the general solution is $v=Ce^{-4\sqrt{2}}t$. We apply the initial condition $v\left(\frac{1}{5}\right)=3.5925=Ce^{-4\sqrt{2}/5}$ and solve to find $11.1378=C$. Therefore, the particular solution is $v(t)=11.1378e^{-4\sqrt{2}t}$.

To determine total displacement we combine the displacement during $0<t<\frac{1}{5}$ and the displacement during $t=\frac{1}{5}$. These calculations produce an improper integral: $x=.4256+\int_{\frac{1}{5}}^{\infty}11.1378e^{-4\sqrt{2}t}dt$. Integration gives the total displacement as 1.06 feet. Since this is far less than the 5 feet to the wall, the cannon will not hit the wall.

Second-order Model

We now model the same situation as second order differential equation with initial values. With this model we use displacement (instead of velocity) as the dependent variable and time as the independent variable. Making the same assumptions as before, we build the following model:

$0<t<\frac{1}{5}$::	$t\geq\frac{1}{5}$

$$\sum F = ma$$

$$F_1 - F_2 = 150a$$

$$4500 - 600\sqrt{2}\frac{dx}{dt} = 150\frac{d^2x}{dt^2}$$

$$30 - 4\sqrt{2}x' = x''$$

$$\sum F = ma$$

$$-F_2 = 150a$$

$$-600\sqrt{2}\frac{dx}{dt} = 150\frac{dx^2}{dt^2}$$

$$-4\sqrt{2}x' = x''$$

$$x'' + 4\sqrt{2}x' = \begin{Bmatrix} 30, & 0 \le t < \frac{1}{5} \\ 0, & t \ge \frac{1}{5} \end{Bmatrix}.$$

We know that at *t=0*, both the displacement and velocity are zero. Thus, the second-order initial value problem is written as:

$$x' + 4\sqrt{2}x' = \begin{cases} 30, & 0 \le t < \frac{1}{5} \\ 0, & t \ge \frac{1}{5} \end{cases}, \; x(0) = 0, \; x'(0) = 0. \qquad (14)$$

<u>Second-order Solution</u>

Here we use the method of Laplace Transforms to solve this initial value problem to determine the function which describes the position of the cannon at any time. We write the right-hand side of the equation using the step function $U(t) = 1, t > 0$ and $= 0, t < 0$. Therefore, $x'' + 4\sqrt{2}\, x' = 30 - 30U(t - \frac{1}{5})$. We convert this to the Laplace space to form an algebraic equation:

$$s^2 X(s) - sx(0) - x'(0) + 4\sqrt{2}\; s\; X(s) + 4\sqrt{2}\; x(0) = \frac{30}{s} - \frac{30}{s}e^{-1/5\,s}$$

$$(s^2 + 4\sqrt{2}s)X(s) = \frac{30}{s} - \frac{30}{s}e^{-s/5}$$

$$X(s) = \frac{30}{s^2(s + 4\sqrt{2})} - \frac{30\, e^{-s/5}}{s^2(s + 4\sqrt{2}s)}$$

Using partial fractions to separate the first term on the right-hand side, we obtain:

$\frac{30}{s^2(s+4\sqrt{2})}=\frac{0.94}{s}+\frac{5.3}{s^2}+\frac{0.94}{s+4\sqrt{2}}$. This substitution produces the following equation in Laplace space:

$$X(s)=-\frac{.94}{s}+\frac{5.3}{s^2}+\frac{.94}{s+4\sqrt{2}}-e^{-s/5}\left(\frac{.94}{s}+\frac{5.3}{s^2}+\frac{.94}{s+4\sqrt{2}}\right)$$

Then, we convert this back to equation space by taking the inverse Laplace transform:

$$x(t)=-.94+5.3t+.94e^{-4\sqrt{2}t}-\mathrm{U}\left(t-\frac{1}{5}\right)\left[-.94+5.3\,(t-\frac{1}{5})+.94e^{-4\sqrt{2}(t-1/5)}\right]$$

We now write the solution as:

$$x(t)=\begin{cases}.94(e^{-4\sqrt{2}t}-1)+5.3t, & 0\le t<\frac{1}{5}\\ 1.1+.94e^{-4\sqrt{2}t}-.94e^{-4\sqrt{2}\,(t-1/5)}, & t\ge\frac{1}{5}\end{cases}$$

Finally, we calculate the total displacement by $t\to\infty\Rightarrow x=1.06$. Therefore, our two solutions (one from a first order model and one found using Laplace Transforms from a second order model) agree.

New Type of Gun System

We continue our historical development of artillery systems. As the size of the battlefield increased, the need for longer-range artillery also increased. To gain longer ranges, the amount of the cannon charge was increased. However, the increasing charges eventually caused cannons to explode. This led to the development of thicker cannon tubes. However, these longer-range cannons had two major drawbacks. First, they were heavy and difficult to move around on the battlefield. Second, after firing, they often moved more than 5 feet and required a large crew to move the cannon hack into battery (the firing position). These problems led to the development of cannons with fixed or stationary carriages and sliding gun tube/breech block assemblies. A damping piston was positioned between the sliding gun tube/breech block assembly and the carriage to absorb the recoil of firing. A recoil spring pushed the tube and breech block back into battery.

Breech-Block Model

Let's develop a model for a heavy artillery howitzer utilizing a sliding gun tube/breech block assembly, recoil spring and damping mechanism. This type of weapon uses a heavy-duty spring and shock absorber. For this howitzer design, the recoil spring has a spring constant $k = 900$ lb/ft. The damping mechanism exerts a force numerically equal to 600 times the instantaneous velocity of the gun tube/breech block assembly. The howitzer is fired at time *t =0* and generates a gas pressure force of 7,500 pounds (nearly double the Parrott Gun). This force lasts for one-fifth of a second and then drops to zero after the projectile has left the tube. The gun tube/breech block assembly weighs 4800 pounds. When the howitzer is fired, the gun tube/breech block assembly recoils rearward and then returns to the starting position.

We want to determine a function which describes the position of the gun tube/breech block assembly for any time, *t*. Our model of the weapon and its recoil system will consist of three parts: the gun tube/breech block assembly, the recoil spring, and the damping mechanism. The gun tube/breech block assembly moves as one component and will be referred to as the breech block. When the propellant is ignited, its combustion produces gases that expand. This expansion creates pressure which forces the projectile along the gun tube, out the muzzle of the weapon and towards its target. Simultaneously, the pressure of the expanding gases acts on the breech block, moving it rearward. The recoil spring retards this rearward motion. Once the projectile has exited the muzzle, the gases are vented and their force upon the breech block dissipates. The restoring force of the recoil spring slows and stops the rearward motion of the breech block and then returns it to the in-battery position. The damping mechanism retards the breech block's movement, assuring it does not 'slam' back into battery. A diagram of the howitzer and its major components is shown in Figure 3.

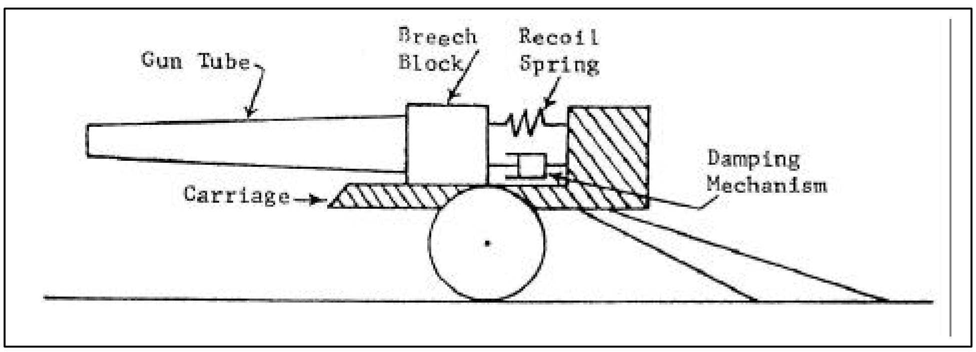


Figure 3: Howitzer and its major component parts.

Model

We will develop and solve a model for this cannon system using differential equations to determine the motion of the breech block and analyze the behavior of this artillery system. For our analysis, we assume that the cannon is horizontal and at rest when the cannon is fired at time *t=0* seconds.

Solution

We use displacement of the breech block *x* as the dependent variable and time *t* as the independent variable. We use Newton's second law to sum the three forces acting on the breech: F_d is the damping force due to the shock absorber; F_s is the force of the spring; and F_b is the force due to the blast of the projectile (lasting only 0.2 seconds). Because of the directions of these forces (against the motion of the breech), we write the following equation: $\sum F = F_b - F_d - F_s$. We now write the corresponding equation term-by-term (here $\dot{x}$ is the derivative of *x* with respect to *t*):

$$\frac{4800}{32}\ddot{x} = 7500 - 600\dot{x} - 900x \ .$$

This equation is valid for $0 < t < 0.2$. Afterward, the forcing function is assumed to be in effect and the model is just the homogeneous part of the equation. Our equation is simplified as $150\ddot{x} + 600\dot{x} + 900x = 7500$, and finally the complete system is written as

$$\begin{aligned} &\ddot{x} + 4\dot{x} + 6x = 50 \qquad && 0 < t < 0.2 \\ &\ddot{x} + 4\dot{x} + 6x = 0 \qquad && t > 0.2 \ . \end{aligned} \tag{15}$$

Using the conjecture that the solution has the form $x(t) = e^{rt}$, the characteristic equation is $r^2 + 4r + 6 = 0$. The roots are $r_{1,2} = -2 \pm \sqrt{2}\, i$. This produces a real variable solution for the homogeneous equation of

$$x_h(t) = c_1\, e^{-2t}\cos(\sqrt{2}\ t) + c_2\, e^{-2t}\sin(\sqrt{2}\ t) \ . \tag{16}$$

A particular solution to the nonhomogeneous equation, which contains the constant term 50, is $x_n(t) = 8.333$. Therefore, the general solution is $x_h(t) = c_1\, e^{-2t}\cos(\sqrt{2}\ t) + c_2\, e^{-2t}\sin(\sqrt{2}\ t) + 8.333$. The initial condition of the breech assembly at rest, $x(0) = 0$ and $\dot{x}(0) = 0$, is satisfied by $c_1 = -8.333$ and $c_2 = -11.785$. Therefore, the particular solution for the time interval $0 < t < 0.2$ is

$$x(t) = -8.333\, e^{-2t}\cos(\sqrt{2}\ t) - 11.785\, e^{-2t}\sin(\sqrt{2}\ t) + 8.333 . \tag{17}$$

In order to solve the second equation of (15) for $t > 0.2$, we need the initial conditions at $t = 0.2$ using Eqn. (17). Substitution gives $x(0.2) = 0.7645$ and $\dot{x}(0.2) = 6.614$. These conditions are substituted into Eqn (16) (general solution to the homogeneous equation) to find the values $c_1 = -1.302$ and $c_2 = 8.567$. Therefore, the particular solution for the time interval $t > 0.2$ is

$$x(t) = -1.302\, e^{-2t}\cos(\sqrt{2}\ t) + 8.567\, e^{-2t}\sin(\sqrt{2}\ t) . \tag{18}$$

We show the graph the equation of motion (Eqns (17) and (18)) in Figure 4 for the time period *0 < t < 3* seconds.

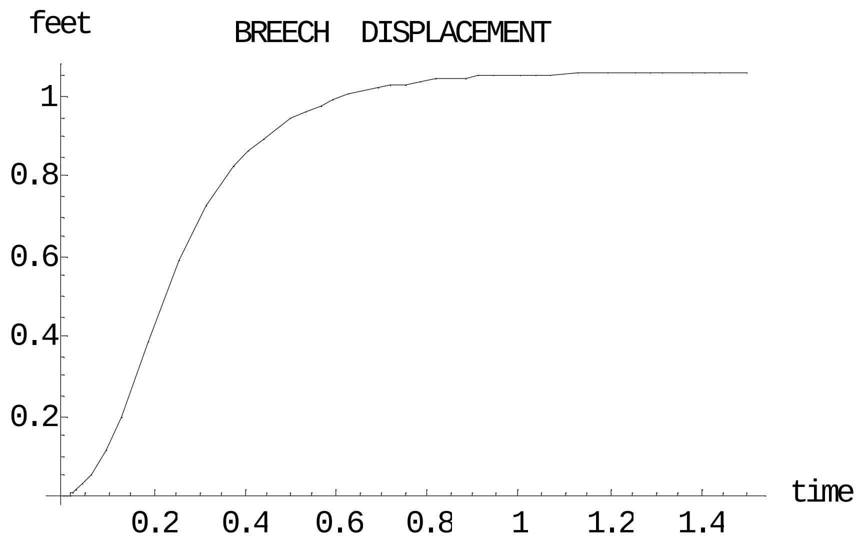
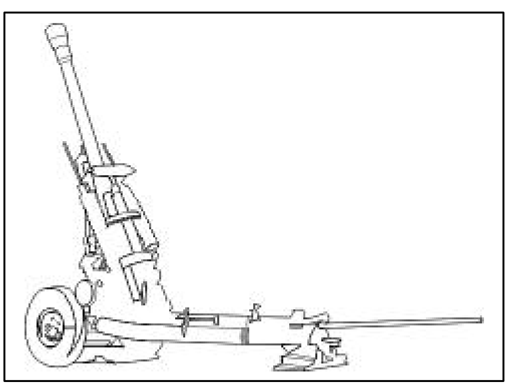


Figure 4: Plots of Eqns (17) and (18) over their respective domains.

It is important to know at what time does the breech block first return to the in-battery position. We do this through calculating the location of the first positive root of Eqn (18). This occurs at $t = 2.328$ seconds.

It is also important to know the maximum displacement of the breech block. We do this by finding the time when the breech block first reaches its maximum displacement. We find the root of the derivative of Eqn (18), $t = 0.5418$, and evaluate Eqn (18) at that point we get $x(0.5418) = 1.692$. This means that the breech will displace a maximum of 1.692 feet (or about 20.3 inches), when the gun fires this projectile.



In the Future

Weapons designers continue to use mathematical modeling and simulations to improve projectiles, breech blocks, damping mechanisms, recoil spring, gun tubes, and muzzle breaks. Currently, many of the new elements of artillery systems are being analyzed using mathematics. These elements include the smart munitions that have been developed for artillery systems, like the sense and destroy armor (SADARM) and the brilliant anti-armor (BAT) munitions. Mathematical modeling was the force multiplier that enabled the control mechanisms of these munitions to produce the accuracy and guidance systems needed. In addition, mathematical models are helping to extend the range and accuracy of “dumb” munitions. Some of the exciting new work ahead for the model builders and artillerists will be to find the optimal employment of these weapons in the form of rockets vs cannons, smart vs dumb munitions, size vs maneuverability of the weapons, and high vs low angle capabilities. Mathematicians will help the artillerists decide these important factors and will attempt to analyze and solve many other artillery issues in the coming years. In the end, the needs of the combat units will be served as more capabilities are developed for artillery systems.

Exercises

1. A projectile is fired over horizontal ground at an initial speed of 1200 ft/sec at an angle of elevation of 30 degrees. Assume there is no air resistance.

a. Write the differential equation model for the projectile’s motion.
b. Where and when does the projectile land?
c. What is the highest point on the trajectory of the round?
d. What is the maximum range of a round fired at that initial speed?

2. Write out the differential equation of motion of a projectile moving at high speeds (assume the resistive force is equal to the square of the velocity). Why is this equation more difficult to solve?

3. A projectile is fired over horizontal ground at an initial speed of 1200 ft/sec at an angle of elevation of 30 degrees. (Same data as exercise 1.) For this model, assume the air resistance is proportional to the velocity with a proportionality constant of k=0.03.

a. Write the differential equation model for the projectile’s motion.
b. Where and when does the projectile land?
c. What is the highest point on the trajectory of the round?

4. Analyze the firing motion of a Parrott Gun with total weight of 4200 lbs, a retarding force of just 300 times the velocity of the gun (the gun is located in sandy soil), and a blast force from the projectile launch of 4600 lbs lasting for 0.15 seconds.

5. Use a spring constant of k=750 lb/ft, assembly weight of 3200 lbs, a damping mechanism of 500 times the velocity of the assembly, and a blast force of 8,000 lbs lasting 0.25 seconds to model a cannon system with a differential equation.

a. Find the equation of motion.
b. Graph the equation of motion for the time period $0 < t < 3$ seconds.

c. At what time does the breech block first return to the in-battery position? What is its instantaneous velocity at this time?
d. At what time does the breech block reach its maximum displacement? What is the maximum displacement?

6. Write an essay to describe how you would analytically find the maximum range of a weapon system using the differential equations in this section. What data would you need to know?

7. What is the difference in the launch mechanism and properties between cannons and rocket systems?

Notes

This "cannon problem" has been used in many forms in many differential equations courses in the Department of Mathematical Sciences for several years. The scenario and solution used for several parts of this article were developed and used by James Hayes, a faculty member in the Department, for the MA 262, applied differential equations course, in 1986.

References

Weapon Systems, United States Army 1998. Washington, DC: Government Printing Office, 1998.

Derek Hart and Tony Croft. *Modelling with Projectiles*. Chichester: Ellis Horwood Limited, 1988.