Modeling in Calculus II

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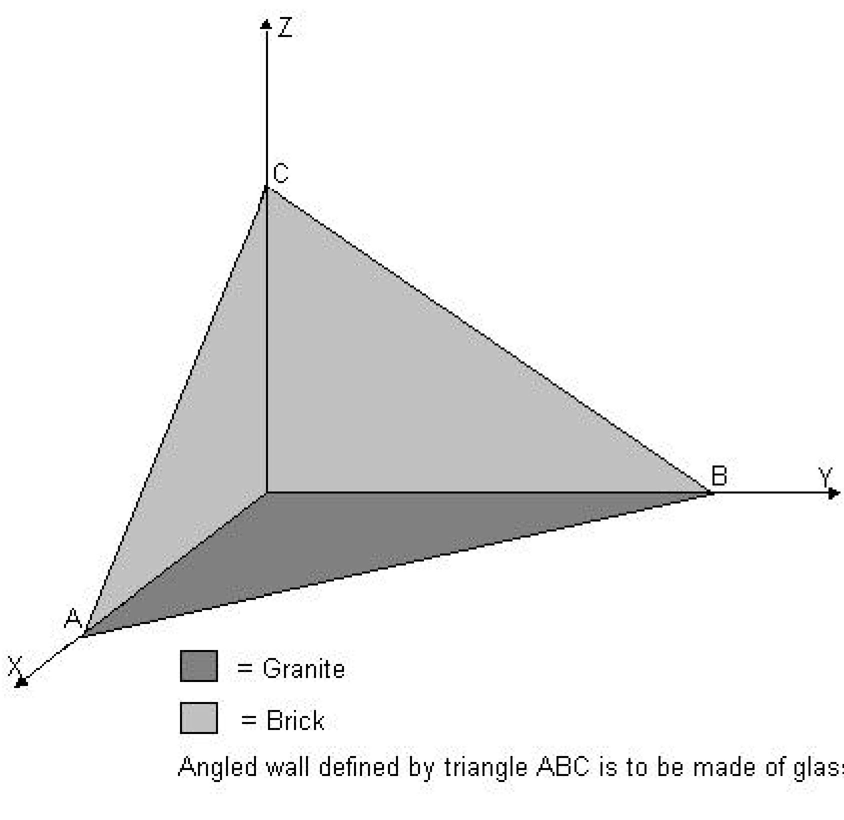
Scenario 1: Optimizing Fiscal Resources

You have been assigned to a district engineer position. The first project your unit receives is to build a visitors center at a recently completed Corps of Engineers dam site. Your commander has examined the budget for completing the visitors center and decided that you can use roughly $100,000 worth of material to complete the outer shell of the building. The architect's concept for the design of the building is that it will be a right tetrahedron. The floor of the tetrahedron is to be constructed out of granite blocks, two of the walls are to be made of brick and one of the walls is to be made of glass. Your commander has asked you to use your knowledge of multiple integrals and optimization to come up with a rough estimate of the maximum volume of a building that can be constructed in the shape of a tetrahedron for $100,000. He provides you with the following sketch and a chart that has the cost of the materials you will use to construct the building.



Angled wall defined by triangle ABC is to be made of glass

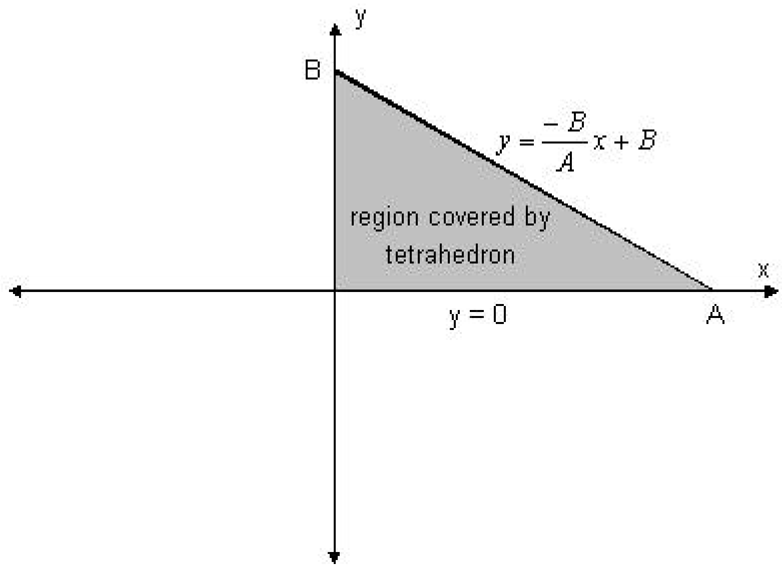
Material	Cost: $ per square meter of surface area
Brick	**3**
Granite	**8**
Glass	**2**

This is your first major project as a district engineer and you are determined to create the best visitors center ever built by a dam site. In order to make a good estimate of the maximum volume for the building, you decide to use the method of Lagrange to optimize the volume of the building subject to the equality constraint of $100,000 worth of building materials.

Problem 1: A good place to begin any optimization problem is with an objective function. In this case, the goal of the project is to optimize the volume of the building. From the sketch you received, the shape of the building is a right tetrahedron. Use a multiple integral to find a generalized function representing the volume of a right tetrahedron.

Solution 1:

A triple integral can be used to find the volume of the tetrahedron. The limits of the inner most integral come from the x-y plane at the bottom of the building and the Plane containing the points A, B, and C which creates the roof of the building. To find the equation of the plane containing A, B, and C, it is necessary to create vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The vectors are then crossed to find a vector normal to the plane. From there, the point normal equation of the plane is used to find the upper limit for the inner most integral. Thus the inner integral limits are $z = 0$ and $z = C - \frac{C}{A}x - \frac{C}{B}y$. The limits on the middle integral come from an examination of the region in the x-y plane covered by the tetrahedron as shown in the graph on the next page.



The lower limit is simply $y = 0$. The upper limit is the line $y = \frac{-B}{A}x + B$. The limits on the outer most integral cover the extreme x values $x = 0$ and $x = A$. Integrating the function $f(x, y, z) = 1$ over these limits yields a generalized formula for the volume of the tetrahedron in terms of the intercepts at points A, B, and C.

$$\int_0^A \int_0^{\frac{-B}{A}x+B} \int_0^{C-\frac{C}{A}x-\frac{C}{B}y} 1\, dz\, dy\, dx = \frac{ABC}{6}$$

Therefore, the objective function for your Lagrange Multiplier Problem is

$$f(A, B, C) = \frac{ABC}{6}$$

Problem 2: You have been restricted to spending $100,000 and you are given material costs in terms of the surface area of each material. By examining the sketch of the building, you can see that the surface of the building is made of four triangles. Two of the triangles are brick, one is glass, and one is granite. Determine a function that represents the cost of the building in terms of the X, Y, and Z intercepts of the tetrahedron. Set this cost function equal to $100,000 to make an equality constraint.

Solution 2:

The cost of the building will be approximated by summing the costs of the four triangles. The cost of each triangle is equal to the surface area of the triangle multiplied by the cost of the triangle's material. The surface area for the triangles made of granite and brick are found using the formula $Area = \frac{1}{2}bh$. The surface area of the triangle made of glass is found by taking half of the magnitude of the cross product of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This method yields the following constraint equation:

$$g(A,B,C) = 4AB + \frac{3}{2}AC + \frac{3}{2}BC + \sqrt{B^2C^2 + A^2C^2 + A^2B^2} = 100000$$

Problem 3: Use the Method of Lagrange with the objective function and equality constraint from steps one and two to determine the largest volume of tetrahedron shaped building that can be constructed for $100,000.

Solution 3:

Using MATHCAD:

$$f(A,B,C) := \frac{A \cdot B \cdot C}{6}$$

$$g(A,B,C) := 4 \cdot A \cdot B + \frac{3}{2} \cdot A \cdot C + \frac{3}{2} \cdot B \cdot C + \sqrt{B^2 \cdot C^2 + A^2 \cdot C^2 + A^2 \cdot B^2}$$

Guess values: $A := 100$ $B := 120$ $C := 150$ $\lambda := 25$

Given

$$\frac{d}{dA}f(A,B,C) = \lambda \cdot \left(\frac{d}{dA}g(A,B,C)\right) \qquad \frac{d}{dC}f(A,B,C) = \lambda \cdot \left(\frac{d}{dC}g(A,B,C)\right)$$

$$\frac{d}{dB}f(A,B,C) = \lambda \cdot \left(\frac{d}{dB}g(A,B,C)\right) \qquad g(A,B,C) = 100000$$

$$\begin{bmatrix} Aval \\ Bval \\ Cval \\ lambda \end{bmatrix} := Find(A, B, C, \lambda)$$

$Aval = 87.706$ $Bval = 87.706$ $Cval = 175.412$ $lambda = 3.373$

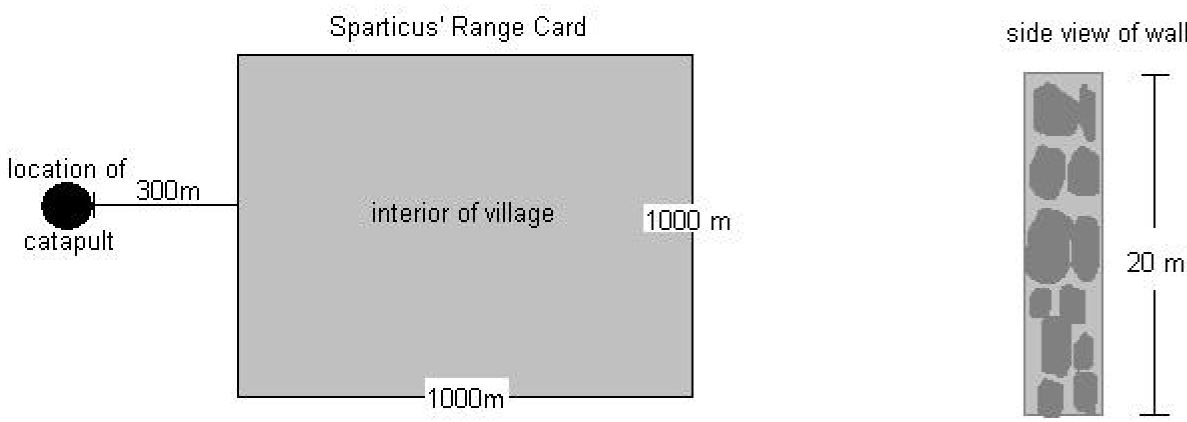
Find Volume: Volume is $\frac{A \cdot B \cdot C}{6}$ $Volume := f(Aval, Bval, Cval)$

$Volume = 2.249 \cdot 10^5$

Scenario 2: A Roman Siege

The year is 27 AD, you and your buddy, Sparticus, are placing a village in Gaul under siege. A stone wall that is 20 meters high surrounds the village. You can get no closer than 300 meters to the wall. Sparticus notices that all of the buildings in the village are made of wood and thinks it would be a good idea to catapult heated rocks over the wall into the village to start it burning and end the siege. You have a catapult that generates an initial velocity of 150m/sec. Sparticus wants you to tell the men at what angle to set the catapult so that the rocks land in the village. Note that the village is perfectly square with each of the four walls being 1000 meters long.

Problem 1: Find the range of angles that you can give to the men to insure success. Concern yourself only with minimum and maximum range. Assume you are centered on one wall and your rock will travel in a direction perpendicular to the wall.



Solution 1:

In its most simple form, this problem is a two-dimensional ballistic motion problem. Begin by deriving vector valued functions that represent the acceleration, velocity, and position of the projectile (rock) at any time t. The equations should be:

$$\vec{a}(t) = \langle 0, \quad -9.8 \rangle$$

$$\vec{v}(t) = \langle 150\cos(J), \quad -9.8t + 150\sin(J) \rangle$$

$$\vec{r}(t) = \langle 150\, t\cos(J), -4.9t^2 + 150\, t\sin(J) \rangle$$

In order to find the angle for the catapult to throw the rock over the near wall of the village, establish a system of equations using the position vector. This system of equations is created by setting the $\hat{i}$ component of the position vector equal to the range of the closest wall (300 meters) and the $\hat{j}$ component equal to the height of the near wall (20 meters). The system of equations that results is:

$$150\,t\cos(J) = 300$$
$$150\,t\sin(J) - 4.9t^2 = 20$$

Solving this system for J yields two angles, 7.585 ^{o} for a low angle shot and 86.22 ^{o} for a high angle shot. The system of equations is only slightly altered to find the angles for the maximum range. The $\hat{i}$ component is increased by 1000 meters and the $\hat{j}$ component is set equal to 0 to make sure the rock lands in the village and not on the back wall. These changes yield the system:

$$150\,t\cos(J) = 1300$$
$$150\,t\sin(J) - 4.9t^2 = 0$$

Solving this system for J also yields two angles, 17.24 ^{o} for a low angle shot and 72.75 ^{o} for a high angle shot.

Therefore the catapult gunners shot restrict their angles to between 7.585 ^{o} and 17.24 ^{o} for low angle shelling of the village. For high angle shelling, the angles should range from 72.75 ^{o} to 86.22 ^{o}.