EDITOR'S NOTES

This semester's issue discusses the impact of technology on our courses. This is an especially timely topic, with the national attention being paid to the role of calculators and computers in mathematics education. For example, there was an interesting discussion in the December *American Mathematical Monthly*. While we are always interested in the views of the larger mathematical community, we learn particularly from each other, so these articles provide helpful insights as we constantly adjust our curricula.

I am always surprised at the different solutions each Department finds to our common problems. There are nice discussions of the use of *Mathematica* and TI calculators from both Navy and Air Force --- while we at USMA use *Mathcad* and the HP48.

Please mark your calendars for the 8th Service Academy Student Mathematics Conference, to be held this year at West Point on 16-19 April. A small article on the topic is in this issue.

I'd like to thank Professor Brad Kline, MAJ Bernie Schliemann, MAJ Mike Shehan, and Professor Joseph Wolcin for their help soliciting articles for this issue. They make this publication possible. I'd also like to thank the Association of Graduates, USMA, for providing the grant for publishing *Mathematica Militaris*.

In the spring issue, we will look at the ways our Departments reach out to our services, DoD, and the nation to provide honest expert advice on quantitative issues.

Best wishes from West Point!

LTC Dave Olwell

CONTENTS

SUBSCRIPTIONS TO *MATHEMATICA MILITARIS*:

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Overview

This issue begins with Professor **Brad Kline**'s overview of how *Mathematica* is used by a variety of instructors at the USAFA. *Mathematica* is not only a Computer Algebra System (CAS) used to support learning at the basic collegiate level, but is now being implemented in higher level mathematics courses at the USAFA. In fact, *Mathematica* is used at both the USCGA and USNA as well. He ends the article by suggesting that "It is up to us to figure out ways in which we can use the technology constructively."

LTC **Dick Jardine** offers a personal account of how he has implemented the use of technology in the classroom. As an advocate and forerunner of using both the calculator and comp uter in the classroom, he clearly identifies the major benefits and pitfalls of their support to learning mathematics.

Probability courses offer a unique challenge. Probability tables, computers, and calculators all offer methods of computing probabilities. At USNA, Professors **John C. Turner** and **Gary O. Fowler** focus Midshipmen on solving interesting probability questions rather than simply computing the associated numerical results. They offer that the calculator is the best method (instead of the traditional tables) to calculate probabilities for a variety of discrete and continuous probability distributions. In addition, they provide an internet address that provides many of the useful probability programs for a variety of calculators.

The next three articles provide some specific insight as to how CAS (specifically *Mathematica*) is used to teach Cadets and Midshipmen at the USAFA and the USCGA. Maj **Deborah Hall** presents an interesting discussion of how the core mathematics courses use *Mathematica* assignments to teach Cadets the utility of learning this powerful program. In addition, she describes how these assigments evolve - beginning with a tutorial on some specific Air Force applications and ending with a project with little guidance as to how the problem should be solved. Finally, Maj Hall outlines the *Mathematica* skills achieved by the end of each core course. Professor **Ernie Manfred** provides an insightful account on the use of CAS at the USCGA. He desribes the use of *Mathematica* Notebooks for laboratory exercises. Furthermore, he presents a discussion concerning the benefits to learning of using this system and its impression on the students. Finally, he provides some important lessons learned. Maj **Tim Cooley** addresses the use of *Mathematica* in the engineering math sequence at the USAFA. His article focuses on using the program for multivariate calculus and differential equation applications. Futhermore, Maj Cooley discusses the integration of *Mathematica* assignments for better "conceptual understanding" of the course material.

The next article was written by the Editor, LTC **Dave Olwell**. He leaves us with the important issue of how to incorporate the recent advancements made in technology into the classroom. Professor **Mark D. Meyerson**'s article about the TI-92 calculator actually provides an excellent discussion as to what students should learn to do "by hand" versus what they should be able to perform with their calculators. He provides some great examples of how the USNA is addressing this interesting issue. Finally, he provides some specific tips for using the TI-92.

Mathematica…It's Not Just For Freshmen Anymore!

Professor Bradford Kline
U S Air Force Academy

Technology has played a prominent role in the so-called "reform" movement in calculus. As computer algebra systems become more affordable and more user-friendly, they are finding their way into more and more calculus curricula across the country. This has certainly been the case at the US Air Force Academy, where *Mathematica* has been an important supplement to core and technical-core courses for a number of years.

However, *Mathematica*'s usefulness to the curriculum need not end at the freshman or sophomore level! *Mathematica* can also serve a purpose in upper-division courses.

At the Air Force Academy, Maj Harry Newton has made extensive use of *Mathematica*'s linear algebra packages in Math 343, a course on computational matrix algebra. Maj Newton uses Mathematica to obtain LU- and QR-factorizations of matrices, and to find eigenvalues through the QR-Algorithm of Francis [2]. In *Mathematica*, the Francis algorithm can be coded in three lines, since so many of the standard matrix operations are already defined:

```
For[i=0,i<100,i++,
{q,r}=N[QRDecomposition[B]];
B=N[r.Transpose[q]]]
```

Maj Rich Schooff also makes use of *Mathematica*'s matrix operations in OR 410, probabilistic models in operations research. Maj Schooff assigns a project to the cadets involving discrete-time Markov models for reliability and safety analysis. In this project, the cadets work with a 6x6 transition probability matrix that contains two unknowns, the failure rate and the coverage factor. With *Mathematica*, one can easily compute large powers of the transition matrix and solve for the reliability of the system.

Lt Col Bill Craine and I use *Mathematica* to demonstrate different geometric properties of complex functions in Math 451, complex variable. The commands CartesianMap and PolarMap in *Mathematica*'s ComplexMap package allow us to plot the image of a rectangle or an arc of a circle under the complex function of our choice. Using these commands, we can show, for example, how the complex exponential maps certain rectangles onto annuli:

```
Needs["Graphics`ComplexMap`"];
Clear[f,z];
f[z_] = E^z;
CartesianMap[f,{-1,1},{0,2 Pi}];
```

The ContourPlot and Plot3D commands can be used with the real and imaginary parts of a complex function to demonstrate numerous properties. For example, by plotting the level curves of the real and imaginary parts of a function on the same set of axes, we can see that the level curves are orthogonal at their intersections and thus get an intuitive understanding of what it means for a map to be "conformal." We can also demonstrate that the real and imaginary parts do not assume local maxima or minima on open sets.

So, *Mathematica* is a great tool for the more computational and visual aspects of our courses. But there is surely not much use for technology in a good ole' fashioned theory course, right? Not so fast! Some schools, such as Appalachian State University [1], are working on incorporating technology into courses such as real analysis—and for theorem proving, no less! A computer algebra system can be used to prove by induction that the sum of the first n natural numbers is $n(n+1)/2$ [1]. Or, it can be used to prove that Simpson's approximation of a definite integral is equivalent to integrating the parabolic approximations of the function on the subintervals.

So, is technology the way of the future for all of our upper-division courses? Some would give an emphatic "No!" Many would argue that it should not be. What is certain, however, is that the technology already exists for us to make significant changes to these courses. It is up to us to figure out the ways in which we can use the technology constructively.

References

1. Bauldry, W. (1996). "Real Analysis with Maple." Presented at the Ninth Annual ICTCM, Reno, NV, 8 Nov 1996.

2. Kincaid, D., & Cheney, W. (1991). Numerical Analysis, Brooks/Cole, 1991, pp. 269-270.

Use Technology Appropriately

LTC Dick Jardine
US Military Academy

This is a confession. I am guilty of the crime of "bludgeoning our students with technology," using the words of Professor Gary Sherman of the

Rose-Hulman Institute of Technology. Gary was a recent invited speaker at USMA, and he used that phrase in describing some of the pitfalls of the calculus reform movement. In addition to being a reform supporter, I have long been an advocate of the use of computer technology in mathematics education. That was the subject of a research paper I wrote as a requirement for an education masters I earned in 1981. While teaching at USMA in the mid-80's, I wrote Pascal programs for the available Apple II and TERAK (can you believe an 8-inch floppy disk!) computers. The programs were used in our probability and statistics courses as electronic blackboards for the display of probability distributions and the associated computation of probabilities. Since my return to USMA in 1994, I have been given responsibility for computers in the Department of Mathematical Sciences, and have been a leading advocate for the use of computers in the mathematics education of cadets. Here are some lessons learned from one who has made just about every mistake there is to be made.

Make perfectly apparent to your students the appropriateness of the use of technology. While giving an class in elementary statistics at a small New England college, I demonstrated the use of the TI-83 to plot the histogram of a data set. After class, a student noted that she could have just as easily done the plot by hand rather than push all the buttons required to generate the graph on the calculator. She was right. I had failed to make clear to the class that now that the data was in the machine, I could not only plot histograms but also do a wide variety of other data analyses with ease, and far faster than anyone could do with paper and pencil. Our students must clearly be shown the advantages that technology offers, rather than see it as a "high-tech" way of doing the same old thing.

Textbook authors often make the same mistake I made. In the current calculus text used at USMA, the authors' first example of the use Euler's method to solve a differential equation is not only done mathematically incorrectly, but also is done on a problem, $\frac{dP}{dt} = P + 1, \quad P(0) = 2$, which our students can solve *exactly* quicker than they can iterate to obtain the numerical approximation. It is not difficult to find an example of a differential equation that the students cannot solve analytically, e.g. $\frac{dP}{dt} = e^{t^2}, \quad P(0) = 2$, but can solve readily using the numerical method. Additionally, a computer algebra system can easily generate a slope field for the latter problem, leading to a complete demonstration of the reform movement's "rule of three": analytical, numerical and graphical solutions of mathematical problems. Require students to generate one slope field by hand and they will quickly appreciate the power of the computer and calculator .

A second lesson learned is to ***be aware of the capability of the student machine***. At USMA, the cadet desktop PC is different than the many flavors of instructor desktops. The software and network configurations are sufficiently different that, in some applications, the students are unable to use applications developed by the faculty. This summer many of our faculty members developed worksheets using *Mathcad* and loaded them onto the academic web server, only to learn after the start of the semester that the server inexplicably prevented cadets from using Netscape to launch the application. Last year, I spent many hours preparing a *Mathcad* worksheet on vector fields that included a photograph of a magnetic field and some important animations. The worksheet worked fine on my 100 MHz pentium with 48 MB of RAM, but would not load on the students' older 486s. Compatibility must always be taken into consideration when developing technology-based learning activities.

A lesson for both students and faculty is that ***every computer algebra system make mistakes***. As an example, the other day my office-mate pointed out that *Mathcad* does not correctly evaluate $\lim_{x\to\infty} \frac{x^{5.7446} - x}{x^{\sqrt{33}}}$. That should be made public information to any pedagogical user of *Mathcad*, if for no other reason than to develop in students the need to become *skeptical interpreters* of the results that are displayed on the screens of their calculators and computers.

We must be careful ***not to overwhelm our students with technology at the expense of their learning mathematics***. Gary Sherman states this lesson better using the analogy, "Do not fall in love with the technological backpack at the expense of the mathematical hike -- mathematics atrophies!" At the start of a recent differential equations course, my students and I spent far more time on how to use *Mathcad* and how to overcome the technical problems of using *Mathcad* on our computer network than we did on the mathematics. In the recent past, our first-year students, many of whom had no experience with a computer, were required to learn not only a computer algebra system, but also how to use a spreadsheet *and* the complicated HP48G calculator (in addition to the academic demands of a very different first-year mathematics course while studying in a novel environment). As a result, the depth of the mathematical learning suffered in the process of learning how to do the mathematics with technology. Most of our first year students can generate eigenvalues and eigenvectors with their calculators; few understand, and less appreciate, the special role they play in mathematical relationships.

Keeping the last lesson in mind***, we owe it to our students to allocate course time to the learning of the technology***. Our students have too many demands on their time to do otherwise. We can develop carefully designed worksheets which aid the learning process, and we can allocate an appropriate amount of class time to the task. Additional instruction can solve some individual difficulties. Ungraded and graded technology-required homework is another alternative. (With the latter, the issue of electronic copying becomes an issue, and the homework should be administered in such a way as to make electronic copying difficult and detectable.) Early in the semester, maximum use should be made of available computer laboratories and computer-equipped classrooms, resources in insufficient supply at USMA.

In addition to allocating time in the course, ***be sure to allocate* plenty *of time to developing worksheets, web pages, and other technology-based learning activities***. Experience reduces the time spent in completing subsequent activities of the same ilk as the previous effort, but many of us have the shared experiences of spending way too much time developing a learning activity that was technology dependent. The time spent must be worth the increment in learning made possible by the technology. And, in addition to the primary technology-based lesson plan, a back-up plan must always be ready when using technology. Invariably the monitor will go out, some battery will die, the projector bulb will blow, or some student will find just the *right* wrong keystroke to push to ensure the computer or calculator demonstration fails. Crafting effective worksheets and web pages are time-consuming, and developing them is an art that is developed with a significant commitment of time on the part of the instructor.

There's an old saying that a turtle never makes progress unless he sticks his neck out. I will continue to try to stay on the leading (hopefully, not "bleeding") edge of the use of technology in the learning of mathematics. My neck is prominently out, as I am always looking for ways to effectively use technology to advance the learning of mathematics. I am guilty of the offense cited by Gary Sherman, but I believe that my students and I have benefited significantly as a result of the mistakes I have made. They have learned that technology is not a panacea, but can be a wonderful partner in solving mathematical problems.

Programmable Calculators in a Probability Course

Professors John C. Turner and Gary O. Fowler
US Naval Academy

All non-technical and some technical majors at the Naval Academy take a core course numbered SM230 and titled "Probability with Naval Applications". This course was added to the core curriculum a few years ago in order to give future naval officers the basic tools of probability. Line officers in the fleet requested this addition to the core curriculum citing the use of probability in tactical decisions, searching methods, and analyzing weapons systems. Students take this course in their second or third year after a three-semester calculus sequence. Since the objective is to teach probability on its own, statistics is not included. While this article describes our experience in this non-technical course, the calculator programs we use to compute probability distributions are also used in the more rigorous course taken by mathematics majors.

Efficient calculation of probability distributions has at least two positive effects. Students in elementary probability courses tend to expend much of their effort calculating and little effort analyzing. By making the calculations more efficient, students can better focus on problem analysis. Simplifying the calculations actually encourages the students to calculate rather than guess. Moreover, simplifying the calculations has the effect of allowing the students to work more problems. Thus increasing their experience and repertoire. Textbooks that support this approach to elementary probability are difficult to find. The course SM230 uses a text we have written and covers five general topics.

1. Set Probability, Conditional Probability, Bayes' Theorem
2. The Binomial Process
3. The Poisson Process
4. Sums of Random Variables and the Normal Distribution
5. The Uniform Distribution, Including Simulation

In any probability course, computing probabilities is fundamental. Devices and methods for these computations include:

- Paper tables
- Computers
- Calculators

Paper tables have several problems. The set of parameter values is limited as a matter of necessity—and are often limited more severely than necessary. Few binomial tables appearing in textbooks include values like 1/3 or 1/6. Hence, it is difficult to calculate probabilities associated with a die or other simple scenarios. The size of many tables is reduced to save space. Of course, the students pay for these savings by performing rather tedious and sometimes complicated algorithms to calculate the missing values. For instance, normal tables are often given only for positive arguments, along with a rule to use for negative arguments and binomial tables frequently contain only values of p smaller that 1/2. We feel learning the needed algorithms to produce the missing values is an unnecessary distraction from the content of the course. Discrete distributions present special problems when the probability mass function (PMF) is tabulated instead of the cumulative distribution function (CDF). It is our opinion that there are few "real" problems that are answered by the PMF and in those rare cases the PMF is easily calculated using the CDF. On the other hand the CDF is needed for many interesting questions and computing it by summing the PMF is unreasonable. Tables of logarithms and trigonometric functions were abandoned long ago. It is time to abandon paper probability tables, too.

There are several options for using computers to calculate probabilities. Spreadsheets contain functions for the common distributions. There are also computer algebra systems, such as *Maple®*, that could be used. Statistics packages or specially written programs are useful in many situations. However, all of these methods are inaccessible when they are most needed: during exams and classroom exercises. In fact the number of students taking the final exam in SM230 is far greater than the number of computers in all the labs at the Naval Academy. Hence, computers are not (currently) a viable tool for calculating probabilities in an elementary course—at least at the Naval Academy.

The calculator option is by far the most attractive. Our students are required to have a suitable calculator, so there is no added expense. They are already familiar with its basic operation by virtue of its use in Calculus. Most importantly, the calculator is available for use during classroom exercises and on tests.

Once the decision was made to use the calculator as a computational tool in SM230, it was necessary to determine the details of the calculator programs and to plan how the programs would be distributed to a few hundred students. Most calculators have either cables or infrared devices for transferring programs. While a quick demonstration of the transfer procedure is sufficient to start the process, our students have proven very adept at linking and transferring programs. Of course, this is a geometric progression and so the calculators are quickly programmed. A benefit to transferring the programs is the near certainty that all the students have the same correct program. For those few calculators that cannot transfer programs, the students must type the program using the calculator's program editor. The final step in the programming process is to check the programs by evaluating the programs at specific values.

We made the decision to keep the programs simple. For the discrete binomial, hypergeometric and Poisson distributions, we simply sum the values of the probability mass function. For the normal distribution we use a rational approximation from Abramowitz and Stegun, Handbook of Mathematical Functions. For example, the binomial

CDF code for a TI85 follows. Code for several calculators is posted at the WWW site *www.nadn.navy.mil/MathDept/courses/fall97/sm230/sm230.htm*.

```
PROGRAM: BCDF
Prompt, P,N,X
sum seq((N nCr M)*(1-P)^(N-
M)*P^M,M,),X,1) STO A
disp A
```

It is not sufficient for our students to calculate the values of CDF's. In fact, this skill alone would be rather useless. Probability calculations are useful only in the context of a problem. We attempt to keep the context simple, while using the context to motivate the questions. For example, we ask a series of questions regarding a shuttle bus. If passengers arrive following a Poisson process and at an average rate of 5 each ten minutes, then what is the probability that an eight-passenger bus leaving in 10 minutes will leave no one standing on the curb because the bus is full? What should the capacity of the bus be in order to be 90% certain that it leaves no one on the curb? If it is an eight-passenger bus, how often should it leave in order to be relatively certain that no one is left on the curb? If it is an eight-passenger bus, what is the largest arrival rate that would mean that usually no one is left on the curb? All these questions can be answered using the Poisson CDF. More importantly the context of the problem suggests reasons for wanting to know the answer. Also, some of the questions require interpretation on the part of the student. The result is that the students must think about what they are doing. The calculation algorithms are no longer the important part of the problem as they were with paper tables. Now the calculation is easy and the question is important. Also observe that the questions asking for the capacity of the bus requires that the student evaluate the inverse Poisson CDF. This is accomplished by trial and error. That is, the student tries a value for the number of arrivals, evaluates the CDF and then adjusts the number of arrivals until the desired probability is achieved. The calculations regarding the parameters are accomplished in a similar manner. We ask questions in context, expect them to make sense and expect that the calculations are not a problem.

We have created a course in which analyzing and solving problems is the central activity. This is accomplished in large part by providing a method of evaluating probability distributions that is as simple as pressing very few buttons on a readily available calculator. The students generally find our approach to this course satisfying. They believe that it is possible to understand the problems and calculate their answers. Most importantly, they learn to analyze problems and calculate answers in situations where probability is an essential element.

CAS in Core Calculus at USAFA

Maj Deborah Hall
US Air Force Academy

Along with the autumn leaves, this fall has brought the implementation of *Mathematica* (version 3.0) into core calculus courses at the US Air Force Academy. Cadets in Math 141 (Calculus I), Math 142 (Calculus II) and Math 152 (Advanced Placed Calculus II) are expected to gain a functional knowledge of *Mathematica* to supplement their calculus knowledge and to help them gain basic syntax skills before entering their upper-level technical courses.

The premise that every entering cadet is computer literate appears untrue. A typical cadet enters USAFA with a cursory computer background (i.e., they know how to surf the web and play high-tech video games). There is still a large group of cadets whose first time sitting at a computer occurs when they receive the machines issued to them. Therefore, it is not surprising that cadets initially find *Mathematica* syntax difficult to understand. Because we, as course directors and instructors, have come to understand the growing pains associated with learning syntax, we start by teaching new cadets basic skills and use a building block approach to further syntax knowledge as it is needed. This need is determined primarily by the progression of calculus topics in each course.

The learning process begins with a computer lab session in each calculus course. Ideally, this lab occurs during the fifth lesson of the initial calculus course taken by a new cadet. During this 50-minute class period, the instructor helps each cadet work through a tutorial notebook containing a set of introductory skills. While sitting at a laboratory computer, cadets are taught how to open

Mathematica, how to apply basic syntax, and how to manipulate basic functions. The skills of executing cells, finding the correct palette, and incorporating textual comments are also covered during this initial encounter with *Mathematica*.

A lesson we learned is that it helps to de-emphasize (in the classroom) *Mathematica* prior to the introductory computer laboratory. A cadet's first reaction to *Mathematica* seems to be an indicator of their future affinity toward this CAS tool. Cadets who try to figure out *Mathematica* on their own (without the necessary skills) sometimes get frustrated and immediately dislike *Mathematica*. Cadets who have a positive, fun experience the first time they are exposed to this tool are much more likely to consider it a worthwhile supplement to their classroom learning.

After the initial computer lab, we leave the tutorial on USAFA's "common drive" - a drive that all cadets can access. This drive includes information provided by course directors to cadets in their course and will be the source of their future *Mathematica* assignments. The cadets may refer back to the tutorial if they are unclear about a given skill, or to cut and paste portions of the text as they need it for future projects.

Henceforth, in each calculus course, cadets are assigned one project per course block (4 blocks in Math 141 and 3 blocks in Math 142). This is where implementation of a "building block" approach begins. For cadets in Math 141, their first project involves basic plotting, entering basic functions, and creating tables. Cadets are asked to perform these *Mathematica* commands within the context of a *Mathematica* assignment notebook. The assignment notebook provides background concerning a hypothetical Air Force application (e.g., the fall 1997 project tasked cadets to assess the long-term health risks for airmen exposed to radioactive strontium-90 during the 1960's). Assignment notebooks are placed on the common drive several lessons before they are due.

For the first three *Mathematica* projects in Math 141, cadets are given a separate, supplemental tutorial that explains any new commands not learned in the computer lab. Cadets may cut, paste and modify these tutorials in order to answer the questions concerning the assigned Air Force application. Cadets are expected to provide their answers in a separate answer notebook that is handed-in for a grade. Answer notebooks serve a dual purpose--they teach cadets how to move between *Mathematica* notebooks and they also reduce the amount of printed paper. This makes the assignment easier for the instructor to grade and it helps preserve the environment!

The last project in Math 141 exposes cadets to the project style they will see in Math 142 (Calculus II). In Math 142, they are no longer provided with a separate tutorial for each project. Instead, the new information is provided within the assignment notebook and is very specific to the assigned task. The intent of this is to "wean" cadets from the expectation that instructors will always provide all background computer information necessary for project completion.

Upon completion of Math 141, cadets can use *Mathematica* to: create plots with labels; create tables; differentiate and integrate functions (both numerically and symbolically); evaluate and graph Riemann sums; and graph and algebraically manipulate vectors. In Math 142, we move away from specific Air Force applications and more into *Mathematica* as a tool for visualizing the types of physical applications cadets are likely to see in core engineering and physics courses at USAFA. Air Force applications are not ignored; we instead incorporate these applications into many coursewide activities since cadets are now more comfortable with the USAFA environment.

By the end of Math 142, cadets have refined their plotting, differentiation and integration skills. Additionally, cadets know how to perform the following Calculus II tasks with *Mathematica*: estimation of area underneath a curve using Trapezoidal and Simpson's rules (including visualization of these methods graphically); application of the Fundamental Theorem of Calculus; computation of the length of a curve; graphing of solids of revolution; computation of volumes; and approximation of functions using Taylor polynomials.

Cadets who are advanced placed into Math 152 learn the same set of skills covered in Math 142, but the learning curve is escalated to provide these skills in one semester instead of two. These cadets are also required to master *Mathematica* skills taught in Math 141. This is difficult, but necessary, for this group, because most cadets in Math 152 continue and take Math 243 (Calculus III). Proficiency of these basic skills is expected.

In addition to these coursewide projects, *Mathematica* has been implemented into our classroom activities (there is a computer in each classroom). This greatly enhances visualization of calculus concepts and reinforces the idea that the computer can be an effective tool for mastering course material. As course directors become more experienced in developing *Mathematica* notebooks, we are able to use *Mathematica* projects as supplements for more time-consuming topics in the syllabus. We sometimes use it to introduce a calculus topic ahead of the classroom lesson. We have had good results with the approach thus far. It helps cadets to realize that there are multiple ways to learn. It is our ultimate goal that this approach will extend their intellectual curiosity beyond the walls of the classroom, encouraging cadets to use computers to check their homework, check answers to course assignments, and begin to use *Mathematica* to explore conceptual questions of interest to them.

The Computer Algebra System in the Mathematics Curriculum at the United States Coast Guard Academy

Professor Ernie Manfred
US Coast Guard Academy

The history of mathematics is replete with outside influences. Currently, graphing calculators, computers and computer algebra systems (CAS) are having an impact on how students are taught mathematics in colleges and universities. The Department of Mathematics at the United States Coast Guard Academy introduced *Mathematica* (CAS) into the calculus sequence beginning in the fall of 1996. This paper will address the following:

i. How is the computer algebra system used in the courses to address major research findings in the learning of mathematics?

ii. Does the use of the CAS raise students' conceptual understanding and problem solving skills?

iii. How does one assess the benefits/impact of a computer algebra system?

i. How does the introduction of a computer algebra system in the calculus sequence address major research findings in the learning of mathematics?

When teaching the calculus more than one approach is used. Three major avenues to introduce topics are:

1. Geometric
2. Analytic
3. Axiomatic

As an example, in teaching vector calculus, we introduce vectors as directed line segments (geometric approach), usually follow that with the n-tuple definition (analytic approach) and lastly describe vectors as objects obeying certain properties (axiomatic approach). Since a CAS performs three basic types of operations:

1. Graphic
2. Numeric
3. Symbolic

it makes sense to use the CAS (geometric $\approx$ graphics, analytic $\approx$ numerical, symbolic $\approx$ axiomatic) to re-enforce the manner in which the calculus is presented. With this as a guide, the department developed laboratory exercises in the form of *Mathematica* Notebooks to be done in a lab format. Current research on how students learn mathematics reveals the obvious: that they learn more efficiently when new information is structured to relate to what they already know. For this reason, lab topics cover the previous week's work. This pattern allows concentration on concepts being taught without the hinderence of manipulative skills. Hopefully, students will understand an idea by using it, i.e. essentially going from meaning to calculation.

ii. Does the use of a CAS raise conceptual understanding and problem-solving skills?

Conceptual understanding and problem solving skills are intimately related. Conceptualization can certainly be enhanced through discoveries encountered in problem solving exercises; and, likewise problem solving skills can be improved as one's understanding of a problem's context is sharpened. Clearly, if one is interested in developing an understanding of basic mathematical principles, one would first cultivate an understanding of the operations of addition, subtraction, etc. However, once these operations are understood, it seems wasteful not to leverage the power of a calculator to perform such operations. Likewise, when discussing the calculus, students must be familiar with the techniques and principles of differentiation and integration. However, once the essence of these operations is understood, the time spent on such subjects can be lessened and much of the required calculation associated with them relegated to the machine. Indeed, the use of a CAS to foster visual imagery reinforces conceptual understanding. This improved conceptualization may then lead to improved problem solving skills. Interestingly, it seems that prior to the introduction of CAS systems, graphing was an application of analysis. Now, with commands like RootFind and the ability to ZOOM, graphing now often provides the keys to meaningful analyses.

iii. How does one assess the use of a CAS in the teaching of the calculus?

At the end of each lab, we ask students to summarize in a written paragraph how the lab has contributed to their understanding the previous week's material. Responses have been very favorable to date. We continually make changes to the labs (from year to year) based on comments from students. Comments from the labs early in the academic year are less favorable than later assignments. As students become more familiar with the syntax, their apprehension decreases, and they become more comfortable with the system.

The introduction of a CAS is analogous to introducing new symbolism, it only causes pain momentarily. The benefits may not be observed or appreciated until it is used in upper-division courses. The introduction of a CAS has not increased the material we teach since it is used primarily to reinforce concepts in the calculus. However, it is anticipated that substantive changes will occur in junior and senior level mathematics and engineering courses. As an example, the computational aspects of linear algebra, numerical analysis and differential equations will be done by the machine allowing more time for concepts, rigor, and exploration. Topics in error analysis and approximation will become the focus of many of the advanced courses (Advanced Engineering Mathematics and Applied Mathematics courses).

The department uses *Mathematica* (version 3.0) for its computer algebra system. All systems have a steep learning curve and ample time must be allowed for students to become familiar with the system. The fact that a CAS is syntax sensitive may help students by providing instant feedback when something is wrong. It certaintly increases attention to detail.

What lessons have we learned?

To justify the expense and effort involved in using a CAS in the calculus sequence, we hope to demonstrate that by **engaging the student in a meaningful learning process, we place greater responsibility for learning on the student.** A major feature of *Mathematica 3.0* is the use of hyperlinks to obtain the commands and necessary syntax to do the lab exercises. These hyperlinks are placed in the lab notebook and give the student access to the instruction manual (via the Help Menu) that not only describes how to format the command but also illustrates the process with examples.

Vital to the success of using a CAS is that the faculty be well trained. It is important to anticipate mistakes in using the system. Including exercises on the lab from the technology section of a textbook, as we do with Anton's text, is important. It links the text and CAS and exposes students to different types of problems that cannot or should not be done by hand. The labs should count for a reasonable portion of the final grade, thereby sending a message that mastering the technology is an important outcome of the course. We have also found that there is a considerable increase in the instructor's workload. Knowledge of the CAS, anticipation of the type of errors students will make, proper preparation of labs, and assessing what students do and say about the labs, are the keys to success in using a computer algebra system.

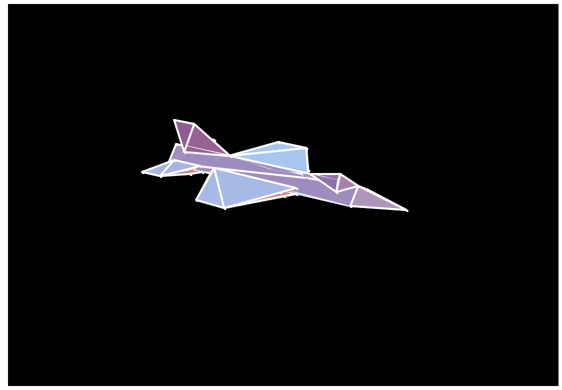
***Mathematica* in the Engineering Division at USAFA**

Maj Tim Cooley
US Air Force Academy

The Engineering Division of the Mathematical Sciences Department at the United States Air Force Academy consists of three main courses: multivariate calculus, introduction to ordinary differential equations and linear algebra, and engineering mathematics. Four years ago we made *Mathematica* the standard computer algebra system (CAS) for use in each of these courses. Last year we revalidated this decision with an extensive look at *Mathematica* versus *Mathcad*. Representatives from all the large mathematics courses as well as most engineering departments and the Physics Department solved sample problems from each of their areas using both software packages. *Mathematica 3.01* was deemed to be the better package for these applications, overall.

The first course in the engineering math sequence for most cadets is multivariate calculus. This course presents a particular challenge in integrating a CAS system since both sophomore and freshman cadets take the course. Because of this, in the fall semester we spend one lesson in the computer laboratory teaching cadets how to use the basics of *Mathematica*. After this training, we place a *Mathematica* notebook on a network drive that the cadets can access. This notebook contains a tutorial which guides cadets through the solutions to several types of problems they may encounter in multivariate calculus. The spring version of this course does not include the computer laboratory lesson, as the freshman cadets learn how to use *Mathematica* in the integral calculus course the previous semester. Our multivariate calculus course is the only engineering math course that has any in-depth *Mathematica* training. In all subsequent courses it is assumed that the cadets have a good solid foundation in *Mathematica* skills.

Currently there are four *Mathematica* assignments in multivariate calculus. The first assignment is very basic. The cadets are led step-by-step through the solution to a problem. They are asked to answer some conceptual questions, but they must do very little actual *Mathematica* coding. Each assignment then becomes progressively more demanding in terms of requiring cadets to solve problems with *Mathematica* without step-by-step guidance. In the last computer assignment, we assign a problem for cadets to solve using *Mathematica* with virtually no guidance. This requires them to have mastered several *Mathematica* commands in the areas of: plotting, solving systems of equations, using vectors, taking derivatives, and performing integration.



In our ordinary differential equations and linear algebra course there are also four computer assignments. In this course, we assume cadets have the basic *Mathematica* skills, so the goal is to use the *Mathematica* assignments to better their conceptual understanding. For instance, in the first assignment, cadets describe the linear transformations that take place as an animated 3-D airplane flies across the screen and performs an in-flight aileron roll. The above figure shows a picture of this airplane. Other assignments have the cadets graphically investigate the solutions of ordinary differential equations using phase plane plots and also explore the solutions of systems of differential equations.

In engineering math we do not have any specific computer assignments. Rather, there are three projects that are assigned and the cadets are encouraged to use *Mathematica* in completing these projects. In most cases, if *Mathematica* is not used the cadets will have significant difficulty finishing the project. At this level, we are trying to instill in the cadets the academic maturity to integrate the use of *Mathematica* into all their course work rather than separate the course into "computer assignments" and "other assignments."

While our strategy for integrating CAS into our engineering math curriculum constantly evolves, we feel the progression described above is the right philosophy. Early in the curriculum we provide a lot of the *Mathematica* code for the cadets. By the end of the curriculum we not only expect them to be able to solve a problem using *Mathematica*, but also to recognize when they need to use a CAS system. Although currently there is only a small amount of data available to determine our success, the initial analysis shows this approach to be successful.

Technology: Horse or Cart?

LTC Dave Olwell
US Military Academy

We mathematics instructors are in the middle of a third wave of technology reform. First there were calculators, and then came personal computers. Now we have the Web. Each wave has shocked our systems of instruction, and affected not only how we teach but what we teach.

Despite its title, the purpose of this note is not to debate whether or not the changes have been for the better. Rather, we shall look at the process of adopting new technologies, and see what lessons we can learn. It is surely true that change will continue, and who knows what the next wave will bring. Can we consolidate the lessons learned from the first three waves, and prepare for the next?

What did the first three waves accomplish? First, they have eliminated the need for routine hand computation and graphing. Second, they have made sharing of ideas and resources very easy. Third, they have expanded the set of problems which can be analyzed by students. These changes are having a profound impact on the teaching and practice of the mathematical sciences.

Yet we struggle with the new technologies. Why? The thesis of this note is that when technology drives reform, we put the cart before the horse.

There is strong anecdotal evidence that technology has been driving reform. Technology has changed **what** our students can do, and **how** they can do it. Technology can not provide a reason **why** they **should.** In other words, we who make choices about curricula are constantly being forced to update our vision of what it means to be an educated person and what the mathematical component of an education should be.

Our vision and technology do not often stay synchronized very long.

We are also handicapped by not having clear understanding of how students learn concepts. This affects our decisions to use technology. For example, students don't need to graph functions by hand anymore --- their calculators and computers will do that task easily. However, it may be that the ability to graph a function by hand is a key component in understanding what a function is, and in retaining the properties of a particular function. If that is the case, then despite technology students should still perform that skill by hand because it improves understanding.

I am saying that there may be skills which are essential for understanding concepts. This is different from saying that there are skills which are essential to be able to do engineering mathematics. We have done a good job at West Point negotiating with our client disciplines what the essential skills are for subsequent engineering course work. I am not sure that anyone has looked at the issue of what skills are necessary for understanding concepts in mathematics courses.

This issue can be illustrated by thinking about the elementary schools. Students are given calculators in early grades, and in many cases they do not learn multiplication tables as their parents did. (I know my son did not.) This causes angst among parents because our vision of what it means to be an educated student includes the ability to multiply in ones' head. This may or may not be an essential skill for future life; the more important open question is this an essential skill for understanding arithmetic and algebra? In other words, is the skill to do multiplication in one's head still a component of what it means to be an educated person? It depends on the vision of the person who sets the curricula.

We should set our goals, and then select technologies that help us implement them. One central goal for students must be to learn the skills that advance understanding of the concepts. Our task is to understand what those skills are.

If we delegate skills to technology simply because we can, without looking at the effects on student understanding, we put the cart before the horse.

The TI-92 Calculator

Professor Mark D. Meyerson
US Naval Academy

This is a "lessons learned" summary of the USNA mathematics department's use of the TI-92 calculator this semester. Several concerns are addressed — pedagogical questions, practical questions, etc. As such this report is an odd mix of history and beliefs. An attempt is made to balance the many differing views held by department members. Many of the ideas are not the author's; they come from various USNA mathematics instructors. The opinions expressed are not official department opinions but were agreed to by all 25 Calculus I instructors. Very specific examples and experiences are included.

Background

Starting with the fall semester, 1997, USNA required each new student to have a calculator that does symbolic computations (such as finding derivatives exactly). At that time only the TI-92 fit the bill. One major motivation for requiring such a calculator was fairness — to insure that no student had an advantage by having a more powerful calculator. Side benefits of a common calculator included that students are more likely to be able to help each other and instructors are more likely to know how to use the student calculator (at least once experienced with it). A possible concern is cost, about $180 vice about $100 for other calculators. But it is cheap compared to computer cost and not that expensive compared to textbook costs.

Based on computer algebra software called *DERIVE*®, the TI-92 has about 70K of available memory and will exactly (symbolically) differentiate and integrate most common functions. It will also solve equations (sometimes exactly, other times approximately), take limits, take partial derivatives, and graph curves and surfaces.

The Big Question

Should we allow calculators in calculus at all, or this one in particular? On the one hand, we need to realistically face the existence of new technology; on the other we're not interested in teaching button pushing. Just as the square root algorithm and log tables have left the curriculum as technology has made them much less important, so parts of the calculus curriculum should be dropped. But also, just as it's still important for elementary school students to learn basic arithmetic (for example, to approximate answers as a check), parts of calculus that the TI-92 does aren't necessarily obsolete. We, as instructors, need to evaluate "What is it that we want our students to learn and how to best teach it in light of this technology?" This last question is what we address in the following.

Striking a Balance

The basic computations (like sin ' = cos or the product rule) still need to be learned, but long involved computations are only of value in drilling basic computations and doing many involved problems is not worthwhile. We recommend a mix of very basic problems that are quicker to do without the calculator but could be done or checked with the calculator and more involved problems that test conceptual understanding for which the calculator doesn't do the whole problem. This semester, it was recommended to all Calculus I instructors that they give at least one "banned calculator" quiz drilling the rules of differentiation. However, on the final exam calculators are to be allowed. There are ways of asking most questions to avoid the calculator if that's desired. Also, insisting that work be shown may suffice. Calculator intensive problems may be more appropriate as projects than test questions.

In Calculus I we should be reviewing the concept of function and teaching the concept of derivative. These are both deep and useful concepts, with many interpretations and applications. Thus we want to teach them in many ways, showing their many facets, and getting students to understand the concepts and to apply them in many different contexts. They should understand that the derivative can be interpreted as the slope of a tangent line, as a limit of a certain algebraic expression (the difference quotient), as a rate of change, and as an operator on functions. They should be able to compute the derivative in many ways (approximately with graphs or data, exactly with algebraic equations). There is a small collection of basic functions that they need to be familiar with, and know the derivatives of (perhaps x^n , *exp*, *ln*, *sin*, *cos*, *tan*, *arcsin*, *arctan*) and basic rules (linearity, product, chain). They should be able to apply the derivative: finding equations of tangent lines, using those equations to approximate functions, comparing rates of change, finding extrema, testing extrema, interpreting the meaning of the derivative, etc.

Since we are only removing the most involved algebraic computations from what we expect students to do, we are not saving a lot of time. Part of the saving goes toward learning to use the calculator. There are lots of other desired uses of the time saved:

a. bringing weaker (including "at risk") students up to a higher level by teaching in different ways.
b. not doing more, but doing what is done better.
c. including deeper mathematics: theory and proofs.
d. including more applications, especially "realistic" ones.

All of these are desirable ways to take advantage of the calculator. We are experimenting with incorporating them. However, with the limited amount of time saved, one shouldn't expect great strides in any of them. Perhaps more time will be saved in Calculus II.

Some Specific Examples of TI-92 Usage

a. There is a "hump" or "learning curve" to get over with the TI-92. There are differences with other calculators that new or casual users will find frustrating. For example, entering "3/7" gives a result of "3/7". To get the decimal approximation one should use the green diamond or put a decimal in the input (3./7). Also, a useful but possibly irritating fact is that many calculations can be done in many ways (e.g., yet another way to get 3/7 in decimal form involves the MODE key).

b. Using the function definition property helps emphasize the input-output nature of functions. It also simplifies other expressions (see next entry) and reduces the likelihood of typos. For example, "define f(x)=2x^2-3" then allows the student to quickly evaluate f(1), f(3), etc. A fruitful activity is to then have students graph f(x), f(x+2), f(x)+2, etc., to see the difference (set y1(x)=f(x), etc.). The STOre key is an even more efficient way to enter functions. This cuts way down on repetitive computations. For example, to get half a dozen secant slopes (approximating a tangent slope) at once, one can enter:

(f(x)-f(5))/(x-5) | x={7,6,5.5,5.1,5.01,5.001} The vertical bar "|" is read as "with".

d. Nice for parametric equations. Makes a subject that students have much trouble with more accessible. In fact, it can reasonably be done in the early weeks as is done with the new Stewart text.

e. To emphasize that the limit is not the same as plugging in a number, "define g(x)=x/x". Then "g(0)" returns "undef"ined, while "limit(g(x),x,0)" returns "1".

f. Deeper understanding of limit concept. The idea of finding the delta for the epsilon used to be too much to ask of our calculus students. Now it's more approachable: there are many paths to it. For example, $lim(1/(x^2+x))=\infty$ as $x \to 0+$. How close does x have to be to 0 to get x bigger than 100? Could use the graph (of function or of function - 100), use solve, do algebra by hand, guess and check, etc.

g. Emphasize implicit differentiation: to view y as a function of x and differentiate $x^2+y^2=4$ with respect to x, one can enter "d(x^2+y(x)^2=4,x)".

h. Showing that the calculator is sometimes wrong gives us more credibility in our insistence on learning concepts. Often the errors that occur are of a sort that are easy to identify for one who understands the ideas.
i. Makes it fairly easy to print graphs in tests in the same format students are use to (with a cable).

j. The "homescreen" can be saved in memory. This can be used to check later as to what went wrong when wrong answers are given, but also can be used to cheat. Sharing of calculators cannot be allowed and consecutive periods should have different tests — even if a student is honest it would be hard to prove that the stored answer was not looked at.

k. Text can be stored. For example "(f(x)-f(a))/(x-a) -> dq" where "->" is the STOre key, then allows typing "dq" to get back the difference quotient. Longer text can be stored as a file using the text editor. Thus, in some sense tests become open book tests.

l. There are programs that can be downloaded or typed in so that the TI-92 will differentiate implicitly and graph implicitly given functions. It's unclear whether it's good pedagogically to use these.

m. It solves equations for realistic problems. No longer does the instructor need to "cook" the problem so that everything comes out even.

n. Instructors need to teach some things with two notations: the standard one and the one used by the calculator. For example, "*d*(cos(*e*^(2x)),x)" may be needed to be written on the board to stress the necessary syntax and notation.

o. The table feature of this calculator can quickly give students a "feel" for numeric data - much like a spreadsheet.

Specific Hints for the TI-92 Operator

a. The mode key can change between radians and degrees. In practice it may be best to stay in radian mode and use 2^{nd} D to get degrees.

b. The graphing aspect ratio can be made correct with zoomdec.

c. The ON key acts as an interrupt (for example, for overlong graphing). Enter acts as a toggle pause.

d. F3 (or clear or enter) is needed to edit functions in the "Y=" screen.

e. In plotting multiple parametric curves, the simultaneous plotting option (from the graph window, under F1-toolbox, format) is particularly elucidating for understanding collisions.

f. Using three green diamond buttons "Y=","graph", and "table" one gets algebraic, graphical(visual), and numeric (3 of the rule of 4) function descriptions very quickly.

g. Composition of functions is somewhat limited: f(g(x)) will give an error message "circular definition" (in newer TI-92's). One can still do f(g(t)) (if f and g are originally defined using x).

h. One sided limits follow the format "limit(f(x),x,a,s)" where for s>0 it's from the right and s<0 it's from the left (both as x goes to a).

i. If the instructor is expecting the student to input complicated function definitions, giving a "check value" (e.g., f(1.6)=3.4145) is useful.

j. Care must be taken in differentiating and exponentiating that the correct versions of *d* and *e* are used.

Some "Errors" That the TI-92 Makes

a. A standard "error" is including nearly vertical segments when plotting step functions or including vertical asymptotes.

b. Another standard "error" is in graphing something like sin(1000x) that has too many oscillations as discussed in section 1.3 of Stewart's new text.

c. The with "|" command seems inconsistent. For example, tan(x)=x has exactly one solution in each interval of the form (kπ/2,(k+2)π/2) for k an odd integer. But "solve(tan(x)=x,x)|x>5π/2 and x<11π/2" gives 1 of the 3 solutions, while after replacing 11 by 9 it finds both of the 2 solutions.

d. limit(sin(1/x)-sin(2/x),x,0) is undefined but comes out 0. Strangely enough, when multiplied, e.g., limit(2(sin(1/x)-sin(2/x)),x,0), it says undefined.

e. There are at least three zeros (a positive one, a negative one, and a two sided one?) that are equal but with unequal reciprocals. For example "limit(x,x,0,1)->b" and "limit(x,x,0,-1)->c" makes "b=c" true, but 1/b is plus infinity, 1/c is minus infinity, and 1/(b+c) is undefined.

f. The inflection point (graph - math menu) mistakenly identifies 3π/2 for 2sin(x)+sin(x)^2.

g. Integrating tan(x) from 0 to π gives 0 instead of undefined.

Service Academy Student Mathematics Conference

The eighth Service Academy Student Mathematics Conference will be held 16-19 April 1998 at West Point, NY. Cadets and Midshipmen will present the results of their research projects in the mathematical sciences.

Attendees will arrive Thursday, 16 April. Cadets and Midshipmen will be billeted in the cadet barracks with hosts from the USMA Mathematics Forum and Pi Mu Epsilon Chapter. Faculty will be billeted at the Hotel Thayer on the grounds of West Point.

Sessions will be on Friday and conclude on Saturday morning. Saturday afternoon and evening will be available for recreational activities, in accordance with the pass policies of each service academy. Transportation to New York City by train is available. Cadets and Midshipmen may depart on Sunday (or Saturday) as their schools see fit.

Abstracts and attendees are requested to USMA by 15 March. A summary of the abstracts will be published in the spring issue of Mathematica Militaris. As last year, we encourage each academy to bring along some second class cadets/midshipmen as observers to prepare them for their own senior year presentations.

LTC Dave Olwell (DSN 688-5987) is the point of contact this year, and he is assisted by MAJ Robbie Williams. Her number is DSN 688-5620.

Try your hand at the USMA problem of the week: *www.dean.usma.edu/math/outreach/pme/potw.htm*