MATHEMATICA MILITARIS

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Editor's Note

Welcome to the late Summer 2011 issue of Mathematica Militaris. We had some noteworthy article submissions this past Spring in response to a general call for papers, and I think you'll enjoy each of them! Also, we pass on our thanks to last year's Editor-in-Chief, Dr. Csilla Szabo, who worked with the authors on the submissions herein.

In order to improve the relevance of this bulletin, we're transitioning this fall to publishing with a more focused call for papers on a current topic of interest across all five of our federally funded service academies. The topics for the next four issues are:

Issue	Topic
No. 20, Vol. 2 (Fall 2011)	Assessing and Remediating Fundamental Skills within the Core Math Program
No. 20, Vol. 3 (Spring 2012)	Beyond the Core Math Program: Electives to Service Engineering Disciplines
No. 21, Vol. 1 (Fall 2012)	Technology in the Probability & Statistics Classroom
No. 21, Vol. 2 (Spring 2013)	Challenging Advanced Students within (or in lieu of) the Core Math Program

As noted in the revised editorial statement (at the end of this issue), *this does not preclude the publication of general submissions in the latter portion of any bulletin, and we welcome contributions that you wish to share with our community*.

We also note that the United States Merchant Marine Academy has been officially incorporated on our new cover and on our distribution list, and we look forward to contributions from the faculty in their Department of Mathematics and Science! For an overview of their department's curricular scope and their faculty's breadth of talent, please visit their website at:

http://www.usmma.edu/academics/departments/mathandscience.shtml.

This year, we will resume publishing the bulletin twice each academic year and have adopted the following publication timelines in order to improve our processes and provide predictability for both contributors and readers.

Event	**Suspense**	
	Fall Semester	**Spring Semester**
Submission deadline to editor	October 10	March 10
Suggested revisions sent to authors	October 20	March 20
Revised articles due to editor	November 1	April 1
Mathematica Militaris published in .pdf	November 10	April 10
Hard copies disseminated to academies	November 20	April 20

Thank you for your past and future contributions, and I look forward to hearing from you!

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Mathematics is a lot more than just STEM (That's just half the story)

Dr. Chris Arney
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Recently, I have been thinking about how mathematics is currently being applied in new ways in information, social, behavioral, and biological science. As a director of Army-related research for five years, I saw that the current problems and issues of Army concern that math could help most, often fell in the areas of information, social, behavioral, and biological sciences (soft or human sciences). The old days of math as partners with just natural and physical scientists and engineers (hard sciences) are long over. Today, applied mathematicians are much more likely to work with information scientists, political scientists, sociologists, psychologists, linguists, or biologists than physicists. And in any case there is much more to mathematics than modeling springs or airplanes and solving differential or algebraic equations. Yet the term STEM (coined by NSF to designate the areas/disciplines of science, technology, engineering and mathematics that it supports through funding), doesn't include many of the areas where mathematicians connect and contribute their work --- language, sociology, anthropology, political science, public policy, cultural studies, cybernetics, history, and many more humanities disciplines. So the bottom line is that while mathematics is definitely an important part of STEM, it connects and collaborates with many areas outside STEM. Let's not restrict ourselves as mathematicians to thinking too often or only about STEM. The rest of the world needs us too and, by the way, we need the rest of the world as well.

I have always had a broad definition of mathematics and I will give some of it here, so I can reference it later. The field of mathematics encompasses many components, perspectives, and branches. "Mathematics is a human activity almost as diverse as the human mind itself."[1] Most people have studied and utilized the *language* of mathematics -- its numbers, algorithms, systems, formulas, equations, processes, and structures to explain many aspects of the quantitative components of daily life. In this form, it is universal and intelligible everywhere and over all time. Another aspect of mathematics is its own scientific structure; it can be envisioned and utilized as the *science* of measurement – finding size, shape, capacity, optimal values, and many more properties of quantitative entities. As an *engineering* (or problem solving) tool, mathematics helps design and build models and algorithms, analyze and solve problems, and understand and decide quantitative issues. In this form it is often referred to as applied mathematics or operations research. Some consider mathematics as a *humanity*, the study of the quantifiable aspects of issues, emotions, problems, and essence of social science, human patterns, thoughts, behaviors, and cultures. Humanistic mathematics is a growing field of interest to mathematicians. Carl Boyer wrote, "Mathematics is an aspect of culture as

[1] Gabor Szego, *Hungarian Problem Book*, Washington: Mathematical Association of America, 1963, p. 6.

well as a collection of algorithms."[2] For some, especially the philosophers and thinkers of ancient times, mathematics is *religion*. It gives its practitioners faith, support, trust, and structure, often looking at the subject through a mystical or magical lens. For others, mathematics is simply a *game* with a set of rules. They do mathematics to experience enjoyable recreation and nothing more, as Gerry Alexanderson wrote, "Mathematics is a world created by the mind of man, and mathematicians are people who devote their lives to what seems to me a wonderful kind of play!"[3] Mathematics can also be a creative *art* – where beauty of its structures, patterns, and processes are espoused, appreciated, developed, and evaluated. "The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds."[4] For me, however, the real essence of mathematics is that it is the connection that links these diverse areas of human endeavor. So mathematics is neither a huge seven-headed monster nor the distillation of all human endeavors. However, mathematics is an important interdisciplinary link between many elements of the human enterprise. It is like a thin line that connects these components and weaves itself around our world. For instance as a language of science, math is always there to provide the grammar, syntax, discourse rules, and notation for science -- yet it is not biology or geology or chemistry or physics. As an engineering tool, math is not always the central way that problems are solved, but it is an element when a quantity or symbolic representation is needed. As an art, mathematics doesn't give any one art form its spirit or meaning, yet it is present in every picture, statue, song, poem, and dance. There is very little art, language, science, religion, engineering, humanity, or game barren of mathematics. Therefore, mathematics is not a huge monster, but more like a delicate goddess that penetrates, touches, and connects many elements in our world and makes our world a better place. Mathematics connects us – our thoughts, our disciplines, our structures, our processes, our lives. Some people, and I hope we are among that group, see these tiny strands of math weaving their way through the world. Through that vision, they see the goddess of mathematics and experience her beauty. A mathematician is anyone who can see those strands and has felt the presence of the goddess' beauty.

Recently, a nice coincidence came my way. I happened to read **The Numerati** by Stephen Baker. While Baker's approach is not technical, it is interesting with myriad examples about mathematical modeling of information, social, and behavioral issues. I liked the way Baker explains how mathematics is changing to embrace the complexity of society's most important and challenging problems:[5] "People with the right smarts can summon meaning from the nearly bottomless sea of data. … The only folks who can make sense of the data we create are crack mathematicians. They know how to turn the bits of our lives into symbols." (p. 6) And Baker explains mathematical modeling in the following manner: Mathematicians "want to calculate for each of us a huge and complex

[2] Carl Boyer, *The Concepts of the Calculus*, New York: Hafner, 1949, preface.
[3] G. L. Alexanderson, "An Interview with Constance Reid," *Two-Year College Math Journal*, Vol. 11, September 1980, p. 238.
[4] John William Navin Sullivan, *Aspects of Science*, New York: A. A. Knopf, 1925.
[5] From here on all the page numbers in parentheses refer to Baker, Stephen, *The Numerati*, New York: Mariner, 2009.

maze of numbers and equations. These are mathematical models. Scientists have been using them for decades. They build them from vast collections of data, with every piece representing a fact or probability. … Building them is painstaking work. " (p. 12) The title of the book comes from the slang Baker uses to describe the new technical elite that control the use of information: "Now these mathematicians and computer scientists are in position to rule the information of our lives. I call them the Numerati." (p. 9)

Baker's book looks at seven perspectives of our lives in chapters entitled: Worker, Shopper, Voter, Blogger, Terrorist, Patient, and Lover. He explains his vision through passages like these: "Data whizzes are pouring into biology, medicine, advertising, sports, politics. They are adding us up. We are being quantified." (p. 7) "We'll be modeled as workers, patients, soldiers, lovers, shoppers, and voters." (p. 12) "The Numerati also want to alter our behavior. If we're shopping, they want us to buy more. At the workplace, they're out to boost our productivity. As patients, they want us healthier and cheaper. … And they attempt to calculate mathematically how to boost our performance." (p. 13-14) "Add all of these efforts together, and we're witnessing (as well as experiencing) the mathematical modeling of humanity. It promises to be one of the great undertakings of the twenty-first century." (p. 13)

I will let Baker say a bit more in his non-technical style through some passages from the book before I conclude with some of my own thoughts. First a little more about what is happening in applying mathematics to our new society:

- "Taken alone, each bit of information is nearly meaningless. But put the bits together, and the patterns describe our tasks and symptoms, our routines at work, the paths we tread through the mall and the supermarket." (p. 4)
- "The exploding world of data, as we'll see, is a giant laboratory of human behavior. It's a test bed for the social sciences, for economic behavior and psychology. … These streams of digital data don't recognize ancient boundaries. They are defined by algorithms, not disciplines." (p. 14)

While Baker isn't like me in seeing networks, network science, complex adaptive systems, system science, and cognitive science (artificial intelligence) as the lynchpins in all this work, he does see the internet as the primary source of information and the environment where all this is happening. He reflects that "the interpretation of our social networks is an exploding field of research, from IBM to terror-trackers at the National Security Agency in Fort Meade, Maryland." (p. 35) He does recognize that this endeavor is complex and that "The Numerati too are grappling with towering complexity. They're looking for patterns in data that describe something almost hopelessly complex: human life and behavior." (p. 201) "And yet, bit by bit, the Numerati make progress. … They learn from their mistakes. They haul in more data. They continue to experiment." (p. 202)

Of course, one of the issues is how to use intelligent (knowledge-based) systems to model data (as language, as text, as quantities). So as we build these models, we are confronting artificial intelligence all over again. Baker doesn't use the terms reductionism or holism, but he sees two paths that are being taken to understand data: "Some have pushed for a logical approach. They follow a tradition pioneered by Aristotle, which divides the entire world of knowledge into vast domains, each with its own facts, rules, and relationships. … trying to build an artificial intelligence that not only knows much of the world's information, but can also make sense of it." (p. 107) "The rival approach … prefers to see the computer as a counting whiz. Statistics are king. Probability defines truth. Speed and counting trump knowledge, and language exists largely as a matrix of numerical relationships." (p. 108) I see a third path that is based in adaptive complexity theory and holistic models, since in my mind the connections are as important as the components. However, that research doesn't get much press although with IBM's computer Watson playing Jeopardy on TV, AI is definitely in the news.

So what does Baker see happening to the world and its citizens as mathematics and information processing become the power commodities:

- "We [non-Numerati] have to understand the methods that produce these analyses, and we must master some of them ourselves." (p. 203)
- "The point is, these statistical tools are going to be quietly assuming more and more power in our lives. We [non-Numerati] might as well learn how to grab the controls and use them for our own interests." (p. 205)
- "Does he [author's son] need to tackle advanced calculus? Should he delve into operations research, learn to manipulate eigenvectors and hidden Markov models? Do he and millions of others need to become Numerati themselves? In a word, no." (p. 206)
- "But the new challenges are different. The Numerati must now predict how we humans will respond to car advertisements or a wage hike. The models they build will fall flat if they fail to understand human behavior." (p. 207)
- "There's plenty of work for anthropologists, linguists, even historians. If there was ever a divide between so-called numbers people and word people, the challenges ahead demolish it." (p. 207)
- "And the rest of us? We should grasp the basics of math and statistics --- certainly better than most of us do today." (p. 214)
- "The mathematicians and computer scientists create magic but only if their formulas contain real, meaningful information from the physical world we inhabit. That's the way it's always been, and even as they mine truckloads of data, it's a team effort." (p. 216)

As you can see by these previous passages, Baker has plenty to say about information, human behavior, mathematics, mathematicians, data, models, and computers. The conclusions in his book are:

- "What's new, of course, is that many of these 'things' the Numerati are busy counting are people. They're adding us up every which way, and they have all of humanity to model." (p. 216)
- "As we encounter mathematical models built to predict our behavior and divine our deepest desires, it is only human for each of us to ask, 'Did they get it right? Is that really me?'" (p. 216)

Let me be clear, I agree with most of what Baker has written, but I don't think he has gone far enough. He has written an interesting story about using the language, science, and engineering of mathematics to harness the information world, especially as it relates to behavioral and social sciences. But mathematics is also changing biology (DNA, genome, models), systems and organizations, artificial intelligence and linguistics, and government and public policy in just as meaningful ways. As a humanity, mathematics is helping us to understand human emotions like cooperation, trust, love, and empathy. And mathematics is playing a key role in our understanding of networks and our building of robots --- two things that soon will completely change our lives. But most of all, mathematics is our means to **Interdisciplinarity**. As my definition stated, mathematics enables us to connect and understand and solve the real important problems of our world (including the military and government). And to be sure, all the important challenges are interdisciplinary – none of the disciplines alone can produce the progress we need.

So what does this mean for mathematics educators like us at the service academies? I think it means we need to follow Baker's advice -- insure all our mathematics students do real modeling and solve problems in interdisciplinary settings. Don't get me wrong, we need to teach concepts, skills and knowledge, but also students should see challenges and issues and data and see mathematics as the connector to the knowledge, the understanding and the solutions. They should grapple with big policy issues like energy, climate change, food security, and health care. They should read the news and talk about issues using mathematics as language, science, engineering, humanity, and art. And we really shouldn't wait until undergraduate education to show students this role of mathematics. Certainly, in elementary and high school math classes, just like our college-level classes, there is time to show students one of math's major roles --- connecting our knowledge and disciplines. Not everyone needs to be a mathematician or study a STEM discipline, but what I'd like to see happen is that every student sees mathematics in a new light --- connecting, cooperating, collaborating, empowering, enabling, supporting, and enjoying.

The History of Mathematics at the Naval Postgraduate School

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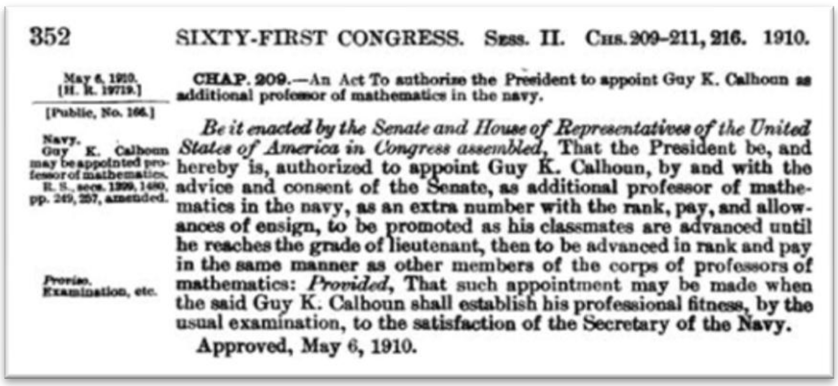
The Early Years

The Department of Applied Mathematics has the longest history of any department at the Naval Postgraduate School and traces its roots to the appointment of Ensign Guy K. Calhoun as Professor of Mathematics in 1910. At the time of his appointment, the school was not yet one year old, having been created only eleven months prior when Secretary of the Navy George von L. Meyer signed Navy General Order No. 27 establishing a School of Marine Engineering at Annapolis. In its first year the program consisted of just ten officer students, with four to eleven years experience since graduation, and was housed in the loft of Isherwood Hall. The original course of studies was two years long and consisted of directed reading by the students under the guidance of the head of the department, a tour on the Bureau of Steam Engineering, and trips to manufacturing plants during the summer. Students soon discovered that their studies were handicapped by inadequate background in mathematics, mechanics, and thermodynamics. The first attempt to meet the need of training in fundamental theory involved creating a series of lectures by distinguished engineers and educators from nearby universities (Columbia, Johns Hopkins, and Harvard to name a few) as well as Washington bureaus. The quality of many of the lecturers and their presentations was very high. Indeed, on April 26, 1912 Dr. Rudolph Diesel lectured on "Diesel Engines" (just a year before his mysterious disappearance.) In spite of the high standard of the lectures (some extending for periods of up to four months), the approach was found wanting as there was insufficient control of the subjects and, not knowing the precise nature of the topic to be lectured on, the students were frequently unable to adequately prepare themselves. And so it was that after just three years, Meyer accepted a proposal to change the school which he formalized on October 31, 1912 when he signed Navy General Order No. 233 which also renamed the school the Postgraduate Department of the Naval Academy. Some months later, in 1913, Ensign Calhoun became the first faculty member assigned to the new Postgraduate Department [1].


352 SIXTY-FIRST CONGRESS. Sess. II. Chs. 209-211, 216. 1910.

May 6, 1910.
[H. R. 19719.]
[Public, No. 166.]

CHAP. 209.—An Act To authorize the President to appoint Guy K. Calhoun as additional professor of mathematics in the navy.

Navy.
Guy K. Calhoun may be appointed professor of mathematics.
R. S., secs. 1399, 1480, pp. 249, 257, amended.

Be it enacted by the Senate and House of Representatives of the United States of America in Congress assembled, That the President be, and hereby is, authorized to appoint Guy K. Calhoun, by and with the advice and consent of the Senate, as additional professor of mathematics in the navy, as an extra number with the rank, pay, and allowances of ensign, to be promoted as his classmates are advanced until he reaches the grade of lieutenant, then to be advanced in rank and pay in the same manner as other members of the corps of professors of mathematics: *Provided*, That such appointment may be made when the said Guy K. Calhoun shall establish his professional fitness, by the usual examination, to the satisfaction of the Secretary of the Navy.

Proviso.
Examination, etc.

Approved, May 6, 1910.

Under the new structure, the school added several new areas of study (gunnery, electrical engineering, radio telegraphy, naval construction, and civil engineering) to that of marine engineering and changed its role to one of providing, in a single year at Annapolis, a common core of coursework of high efficiency for all groups in the fundamentals of engineering which served as preparation for students who would later complete their studies at other universities or installations. For example, after the first year in the Postgraduate Department, students in Electrical Engineering would complete their studies by spending one additional year at Columbia University where they would complete an M.S. degree. Those in Naval Construction would spend seven months at the Postgraduate department and complete their studies by spending two more years at MIT leading to an M.S. degree. Whereas, students in Ordnance would spend only four months at Annapolis and then go on to six months at the Naval Proving Ground, followed by four months at a steel plant, followed by four months at the Naval Gun Factory, followed by one month at Bausch & Lomb (Rochester, NY), finally followed by another month at Sperry Gyroscope Co. (Brooklyn, NY) with four additional months of specialization at one of the foregoing locations.

Foundations: Professor Ralph E. Root

In 1913, Prof. Ralph E. Root was hired as Professor of Mathematics and Mechanics at the Naval Academy. He had earned his Ph.D. at the University of Chicago in 1911 under the direction of the illustrious Prof. E.H. Moore whose list of doctoral students includes such notables as George Birkhoff, Leonard Dickson, Theophil Hildebrandt, R.L. Moore, and Oswald Veblen. Indeed, Root's dissertation work was so fundamental to the early development of the concept of 'neighborhood' that it encompasses an entire section in Aull and Lowen's Handbook of the History of General Topology [2].

This was a very early period of development for the Postgraduate Department which had been officially formed just months before and was only beginning to organize itself under LCMDR J.P. Morton. Prof. Root's talents and inclinations quickly lead him to become involved in the budding program and on February 1, 1914 he became the first civilian faculty member to join the Postgraduate Department [1] when he was hired to take charge of the postgraduate course in mathematics and mechanics. He was later joined by Prof. Leonard A. Doggett who would become the first Professor of Electrical Engineering. Doggett had received his M.E.E. from Harvard in 1910 and had been at Cambridge pursuing further study for the three years following. In September of 1916 they were joined by Prof. H.A. Everett who had been hired away from MIT by RADM John Halligan (then Commander) to become the first Professor of Marine Engineering [3]. It is curious that in 1916, although Root was the only professor with a PhD, Everett, holding only a B.S., was the highest paid at a princely $3,500 per annum presumably on the basis of his many more years of experience as he had been a professor at MIT since his graduation in 1903. Both Root and Doggett earned a mere $3,000 per annum. At that time, Prof. Paul J. Dashiell USN, who was a Professor in the Mathematics Department of the Naval Academy, taught the requisite courses in chemistry. None of the three Postgraduate Department professors had any duties in connection with the undergraduate work at the Naval Academy.

With the declaration of war in April, 1917, the work of the Postgraduate Department was interrupted as its commander and student officers were immediately ordered to other duty, and the teaching force was assigned to appropriate departments of the Naval Academy. Some two years later, with the Armistice signed and the war ended, the Postgraduate Department reopened with its own building (the old Marine Barracks - named Halligan Hall) in June 1919. Fleet Admiral E.J. King, then a Captain, was head of the department. Only Professors Root and Doggett remained from original faculty [4]. Prof. Everett had left during the war and went on to a long and distinguished career at the University of Pennsylvania where he would be joined by Prof. Doggett some years later. Captain King quickly hired a new Professor of Mechanical Engineering (William D. Ennis, M.E.), and three Assistant Professors, one in each of the three divisions of the department - Mathematics and Mechanics, Electrical Engineering and Physics, and Mechanical Engineering. For Mathematics and Mechanics this new faculty member was Prof. C.C. Bramble, who had been at Annapolis as an instructor in the Department of Mathematics since 1917. At that date, Bramble and Root were still the only two faculty members holding the PhD. They would be joined in 1922 by C.H. Rawlins.

Captain King made many organizational improvements during his two year tenure which culminated in the institution officially becoming the Naval Postgraduate School. He added a Professor of Chemistry and Metals, and arranged for a number of officer assistants to be detailed to the school for administrative duties. He also established curricula for mechanical, electrical, radio, and aeronautical engineering, as well as ordnance, naval construction, and civil engineering. The courses of study for the four engineering curricula were identical for the first year; at the end of that time the students would begin their specialization at several other institutions. The remaining three curricula did not fit in quite as conveniently but spent most of the first year at Annapolis before going on to other institutions to conclude their studies.

In 1931 the Postgraduate School had fifteen faculty – four in Mathematics and Mechanics (C.C. Bramble, W.M. Coates, C.H. Rawlins, and R.E. Root), three in Mechanical Engineering, three in Electrical Engineering, two in Metallurgy and Chemistry, one in Physics, one in Radio, and one in Modern Languages [4]. W.M. Coates had been hired in 1931, apparently to replace P.E. Hemke who left earlier in the year to join the faculty at the Case School of Applied Science (he would go to RPI a few years later). Although Coates had done his dissertation research in 1929 on thin walled pressure vessels as a student of Timoshenko at Michigan, he had, in 1924, been at Ludwig Prandtl's institute in Goettingen working on aerodynamics. In those days, spending time with Prandtl was an essential step for anyone wishing to work in the field of aerodynamics and it appears that with these bona fides he was hired specifically to present the aerodynamics lectures previously given by Hemke. He would become the founding chairman of the Department of Aeronautics when it was formed in 1946 which makes that department the first of several that trace their origins to the Department of Mathematics and Mechanics.

In April 1944, just months before D-day, Captain Herman A. Spanagel was appointed Head of the Postgraduate School, a position he would hold for six years. During his first few years he and Root worked together to lay much of the foundation for the current structure of the school. These were difficult days and the burdens on Root and the few remaining civilian faculty must have been great (by 1943 only five were not in uniform [4]). However, the first of Root and Spanagel's efforts did come to fruition when the school received congressional authority to grant academic degrees in 1945. This was soon followed by a reorganization in 1946 that created, for the first time, traditional academic departments, and later, in 1947, by the creation of the position of a civilian Academic Dean.

Prof. and Mrs. Ralph Root at the dedication of Root Hall in 1958.

Even after his retirement in 1946, Root remained a powerful force and towering figure in the school. He continued to be listed as "Senior Professor of Mathematics" in a place of honor in the academic catalog just below the entry for the Academic Dean until 1953, and he was the first faculty member to be given the title of Emeritus Professor. Although he did not come west with the school during the cross-country move in 1951, he did come out to Monterey in 1958 with his wife Mary to attend the dedication of Root Hall, named in his honor. He was the first civilian to ever be honored in this way at NPS.

Evolutions: Professor W. Randolph Church

The Department of Mathematics and Mechanics was one of the seven original academic departments created during Spanagel's reorganization, and Prof. W. Randolph Church was appointed as chairman with the retirement of Prof. Root in 1946. Professor Church, who had joined the school in 1938 as a civilian and then spent the war years on active duty in the Navy, was a keen student of new developments in applied mathematics. The applications of mathematics and statistics to strategy in anti-submarine warfare had led to many other applications in the analysis of naval operations. Fleet Admiral E.J. King, who had been head of the Postgraduate Department following WWI, was also well aware of the power of applying quantitative analysis. In his report [5] he noted that such analysis "sometimes increased the effectiveness of weapons by factors of three or five."


Lieutenant Commander W. Randolph Church meets with Admiral Hyman Rickover in 1946.

In 1950, the CNO ADM Forrest P. Sherman tasked Superintendent RADM E.E. Herrmann to set up a twelve month curriculum in Operations Research at an appropriate civilian institution. Unable to find any civilian institution that was willing to stand up such a program, Herrmann and OEG director Dr. Jacinto Steinhardt proposed to establish a six-term curriculum at the Postgraduate School [6]. This proposal was eventually approved and the first group of officers entered the curriculum in August, 1951. Professor Charles H. Torrance from Mathematics and Mechanics (who had spent 1944-45 working at the Bureau of Ordnance) and Professor W. Peyton Cunningham from Physics taught the initial courses in this discipline. They were soon assisted by Professor Tom Oberbeck, who joined the Mathematics and Mechanics faculty in 1951 after having worked on submarine warfare at the Operations Evaluation Group (OEG) for a number of years after the war. There was very little open literature on the subject of Operations Research at the time (only the unclassified version of Morse and Kimball's 1946 report *Methods of Operations Research* that had been published one year earlier [7]), and hence the rather unique NPS ability to use classified reports from the OEG and other sources was critical in the development of the program of instruction. After a period of growth, during which several statisticians were added to the faculty to handle the gradual shift in emphasis from physical science to statistical analysis as the curriculum adjusted to the needs of the Navy, the School created the Department of Operations Research with Oberbeck as Chairman in 1961. He was succeeded, three years later, by Jack Borsting, who was also from the Department of Mathematics and Mechanics and would later go on to serve as Provost.

In 1966, the separation of Operations Research from Mathematics and Mechanics was completed with the transfer of statistics to the Department of Operations Research along with five of the professors who covered this subject. At this time the offspring to which Mathematics and Mechanics had given birth had nineteen professors and was still growing rapidly.

Professor Church was also keenly interested in the development of computers, started in the war years by the Harvard Mark I and by the University of Pennsylvania Moore School Computer. Indeed, the Department of Mathematics and Mechanics had already played a pivotal role in the history of computing by that point, as it was our own Prof. C.C. Bramble who had recommended the development of the Harvard Mark II for the Naval Proving Ground at Dahlgren. Howard Aiken, computer pioneer and developer of the Mark II, recalls it this way in an interview [8] in February, 1973:

> *"Mark II was built for the Naval Proving Ground at Dahlgren on the recommendation of Professor Clinton Bramble of the United States Naval Academy, who was a mathematician and who was on duty as a Naval Officer. And Bramble was able to foresee that they had to quit this hand stuff in the making of range tables. That's why we built the computer. And Albert Worthheimer found the money for it and signed the contract.*
>
> *In November of 1944, the Bureau of Ordnance requested the Computation Laboratory, then operating as a naval activity, undertake the design and construction of an automatic digital calculator for installation at the Naval Proving Ground."*

It is interesting to note the Prof. Bramble's first contacts with Dahlgren were in 1924. He notes in a January 1977 interview [9] that:

> *"In those days, there was no bridge across the Potomac. I used to call up, and they'd send a boat over to Morgantown, Maryland, for me. When I came down, it was just for general interest in ordnance problems while I was teaching ordnance courses at the Naval Postgraduate School. The courses included ballistics and gun design, both exterior and interior ballistics.*
>
> *Naturally I was interested in the current problems in those areas, so periodically I would get in touch with Dr. Thompson, who was at that time the Senior Scientist at Dahlgren. It was a very informal contact, but that was my way of maintaining a live interest in current ordnance problems and the research that was going on. I also did the same sort of thing with the Army Proving Ground at Aberdeen.*
>
> *When the national emergency [World War II] came on and the decision was made to move the ballistics work out of Washington from the Bureau of Ordnance to Dahlgren, the Postgraduate School was requested to transfer me to Dahlgren, but the Head of the Postgraduate School wouldn't agree, so they compromised by sending me to Dahlgren 4 days a week. That was the beginning of the ballistic work and the beginning of the Computation Laboratory because, at that time, there were only two mathematicians employed at Dahlgren. They*

were at about a GS-7 or GS-9 level. That was back about 1942, and there were also a couple of women at the GS-5 level."

Although he split his time between the Postgraduate School and Dahlgren for several years at that point, Prof. Bramble eventually moved to Dahlgren full-time in 1947 when he was appointed Head of the Computation and Ballistics Department. In 1951 he was selected as Dahlgren's first Director of Research, a position he held until his retirement in January 1954.

Although Professor W. E. Bleick, 1946, and B. J. Lockhart, 1948, had also been involved with these computer developments before coming to the Department of Mathematics and Mechanics, it was Professor Church who led the movement to obtain the first electronic automatic digital computer. And so it was that in 1953, an NCR 102A, was hoisted by a crane through a second floor window in Root Hall and installed in the Mathematics and Mechanics Department. This precursor machine, as well as the development of its use in instruction and research, resulted in the acquisition in 1960 of the world's first all solid-state computer – the CDC 1604 Model 1, Serial No. 1 – which was designed, built, and personally certified in the lobby of Spanagel Hall by the legendary Seymour Cray. This was the first of ten such machines, ordered by the Navy's Bureau of Ships for its Operational Control Centers. The installation of the CDC 1604 coincided with the formation of the School's Computer Facility. Computer courses quickly became standard in almost every curriculum, and the use of the computer in research work increased rapidly at the School. In 1969 the Computer Center was moved to the newly constructed Ingersoll Hall which was the first building on campus constructed specifically to accommodate computers. The Department of Mathematics (which had been officially renamed in 1966) also moved to Ingersoll Hall that same year.

The Era of Computing

In 1966 Prof. Church stepped down as chairman following twenty years of impeccable leadership. Although his legacy lives on to this day, his life would end tragically, on February 7, 1969. In December of that year, the school's computer facility was renamed the W.R. Church Computer Center in his honor. The following year, Prof. Doug Williams, of the Department of Mathematics was appointed the founding director (a position he would hold until his retirement in 1994). He had originally come to the Department from Scotland on sabbatical as a Visiting Professor in 1961 and then stayed on as a regular faculty member when he took over as director of the school's computing facilities in 1962 (which had previously been directed by Prof. Elmo Stewart, also of the Department of Mathematics). Prof. Williams started the very first curricula centered on computing when he created the Management Data Processing curriculum in 1963. He also

Prof. Doug Williams gives a tour to visiting Chinese officers on October 26, 1964.

started Computer Science curriculum in 1967 and served as Academic Associate for both curricula until Prof. Uno Kodres took over for him in 1972.

By 1967 computers had become critical to almost every technical field, and in that same year Computer Science courses were first listed under a separate heading in the catalog of offerings from the Department of Mathematics. The department quickly began adding more faculty in this area. Two years later, Gary Kildall joined the department as an instructor of mathematics to fulfill his draft obligation to the US Navy. His pioneering work during his years as part of the department fundamentally changed the nature of computing, particularly his work on dataflow analysis, and his creation of PL/M (the first high-level language developed for microprocessors) and CP/M (the first operating system for microcomputers).

Eventually, the existence of a group of computing specialists within the Department of Mathematics and their interaction with faculty in other departments (chiefly Electrical Engineering and Operations Research) who worked with computers led to formation of the Computer Science Group in 1973. The group was first chaired by Prof. Gerry Barksdale and later by Prof. Uno Kodres, both from the Department of Mathematics. All of the professors involved in the group maintained their status in their home departments until 1977 when the Department of Computer Science was formed. At that time, five faculty members moved from Mathematics to the new department and CDR Charles Gibfried was appointed chairman.

Department Chairmen

1914-1945	*Ralph E. Root*
1946-1965	*W. Randolph Church*
1966-1971	*Robert E. Gaskell*
1972-1973	*W. Max Woods*
1974-1975	*Ladis D. Kovach*
1976-1983	*Carroll O. Wilde*
1984-1986	*Gordon E. Latta*
1986-1992	*Harold M. Fredricksen*
1993-1996	*Richard H. Franke*
1996	*Guillermo Owen*
1997-1998	*W. Max Woods*
1999-2002	*Michael A. Morgan*
2003-2008	*Clyde L. Scandrett*
2009-	*Carlos F. Borges*

Thus, in about thirty years, the Department of Mathematics had seen two sub-disciplines emerge and develop into thriving departments, each with its own cadre of graduate students, student thesis effort, and sponsored research.

Following the separation of Computer Science from Mathematics, the department saw a ten year period where it was functioning once again almost exclusively as a service department. The Mathematics curriculum (380), which had been established in 1956, was officially decommissioned in 1976 acting on a recommendation from the Navy Graduate Education Program Select Study Committee in 1975 [10]. This committee recommended the elimination of three curricula from NPS and the redirection of future students in those curricula to programs at civilian universities. The three affected curricula were Mathematics (380), Physics (381), and Chemistry (382). These changes left both the Mathematics and Physics departments intact, because of their involvement in other curricula, but resulted in the closure of the Chemistry Department. The Department of Mathematics maintained its degree granting authority, but without an official curriculum, only a handful of students received the MS in Mathematics between 1976 and 1986; most of these were dual majors with Operations Research or Computer Science.

The ONR Mathematics Research Chair

In September of 1979, ONR established a Mathematics Research Chair within the department with the objective of providing a stronger interaction between ONR and NPS with regard to mathematical research of particular interest to the Navy, as well as to stimulate professional development of students and faculty in the areas of expertise represented by the incumbent. It was further envisioned that the incumbent chair would be exposed to problems that are important to the Navy and would be expected to make contributions toward their solution while at NPS and for an extended period after the period of residence. The first incumbent to the position was Professor Gordon E. Latta. During his two years in the position, Professor Latta carried out a variety of research activities related to the Chair's objectives. And although no papers or technical reports were published in connection with his incumbency; many of his results were disseminated, primarily through interaction with colleagues at NPS and presentations at professional meetings. His presence also led to the development of the mathematics microprocessor laboratory. At the end of his one year term, Prof. Latta joined the regular faculty of the department and would become department chairman just two years later.

Although the ONR Chair sat vacant in 1982, it was continued in 1983 with Professor Garrett Birkhoff's arrival in January of that year. In connection with Prof. Birkhoff's incumbency an international conference on elliptic problem solvers was held at NPS also under ONR sponsorship. This conference attracted a virtual who's who of researchers in the field and the proceedings were published under the title "*Elliptic Problem Solvers II*" the following year by Academic Press, jointly edited by Prof. Birkhoff and NPS Prof. Arthur Schoenstadt. Prof. Birkhoff also gave a series of lectures on computational fluid dynamics during his incumbency that helped initiate a number of interdisciplinary efforts across the campus. Unfortunately, the ONR Chair program was not renewed in 1984.

Renewals and Reversals

The 380 curriculum was reestablished in 1987 following an external review of the department. This initiated a strong period of growth and renewal in the department. More than half of the current faculty were recruited in the seven year period following the reestablishment of the curriculum. Throughout the 1990s, the department graduated an average of roughly six students per year with a steady mix of inputs from the Navy, Army, and Marine Corps.

During this period of growth the department reorganized into the three research focus areas we still have today - Applied Analysis, Numerical Analysis/Scientific Computing, and Discrete Mathematics. Although many faculty have interests that overlap more than one of these areas, we are currently grouped roughly as follows:

Applied Analysis	Numerical Analysis / Scientific Computing	Discrete Mathematics
– Donald A. Danielson	– Carlos F. Borges	– David R. Canright
– Christopher L. Frenzen	– Fariba Fahroo	– Harold Fredricksen
– Wei Kang	– Francis X. Giraldo	– Ralucca Gera
– Arthur J. Krener	– William B. Gragg	– Craig W. Rasmussen
– Guillermo Owen	– Beny Neta	– Pantelimon Stanica
– Clyde L. Scandrett	– Hong Zhou	

In early 2001, the NPS superintendent officially deactivated the 380 curriculum for the second time, ostensibly due to low Navy enrollments. Although the department maintained its degree granting authority, other services were not allowed to matriculate students in Applied Mathematics during the remainder of the superintendent's term, and this led to the loss of student inputs from the Army and Marine Corps. Upon his departure, the new superintendent, RADM Wells, officially changed the enrollment policy and allowed other services to once more matriculate in Applied Mathematics. Since then, the department has been able to begin rebuilding our program, and in the last three years we have started to produce a small but steady stream of graduates. At the current time, about half of our graduates are US Army students who come here to prepare for a teaching assignment at the United States Military Academy at West Point. Most of the others are dual majors who are enrolled in other NPS curricula, primarily from Operations Research and Computer Science.

Acknowledgements

The author wishes to express his deepest gratitude to John Sanders (Special Collections Manager) and Greta Marlatt (Supervisory Librarian) of the Dudley Knox Library for all of their help in putting together this history. I would also like to thank my friends and colleagues, Prof. Carroll Wilde, Prof. Bob Read, and Prof. Doug Williams for many valuable conversations and email exchanges that helped me fill in some of the blanks. Finally, I would like to dedicate this work to Prof. Warren Randolph Church, I never got to meet you but you have become a giant in my eyes.

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Mathematics Department Internships at the USNA

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The Naval Academy enables high-performing midshipman to spend several weeks during the summer enhancing their undergraduate education with experience at cultural, policy and research institutions. For periods of nominally two to four weeks, these students obtain valuable insight into the methods of field and laboratory research, which hopefully gives these rising officers greater options in their Capstone projects here at USNA, a wider appreciation of cultures and governing issues beyond the military, and the motivation to continue research and education after they graduate.

Each fall, interested faculties identify potential internship opportunities, often at institutions where the faculty member is collaborating on active research projects. Once approved by Academy leadership, these proposed internships are advertised to the student body in the spring and assignments are made prior to the end of that term. Sponsoring faculty members prepare their midshipmen logistically and academically, but in only a few cases actually accompany the student on the internship. The opportunities vary widely – from work at traditional military research labs and scientific universities sponsored by the divisions of Math and Science or Engineering and Weapons, to more cultural experiences sponsored through the Humanities and Social Sciences Division, the Leadership, Ethics and Law Department, or the International Programs Office. Examples in the humanities areas include leadership internships at other colleges and preparatory schools, the Language Studies Abroad Program, and the Language, Regional Experience and Culture program. Most internships are open to any midshipman regardless of major.

These summer internships are generally conducted on what would otherwise be the student's summer leave time, and so are voluntary. Yet even with the other mandatory summer training requirements (e.g. fleet cruises) there is widespread participation in the internship program. There are two instances where summer internships are required – those students participating in the Bowman Scholarship program (superior performers heading to nuclear power training), and those rising 1/C midshipmen who are competitive for medical school billets after graduation. The subjects and locations for these internship programs are specific to the midshipman's graduate education plans.

The Naval Academy internship program is coordinated through the Academy's Research Office, with assistance from coordinators in each academic department and division. These coordinators help maintain logistical, fiscal and academic accountability on the hundreds of students spread globally each summer due to these internships.

Upcoming Mathematics Department Midshipman Summer Internships recently approved by the Superintendent include:

1. Chief of Naval Operations Assessment Division (OPNAV-N81), intended for Math, Applied Math, and Quantitative Economics majors interested in Operations Analysis.
2. Chief of Naval Operations Strategic Planning and Analysis (OPNAV-N13), also intended for Math, Applied Math, and Quantitative Economics majors interested in Operations Analysis.
3. Naval Research Laboratory Advanced Information Technology Branch (NRL-AIT), intended for midshipmen in technical majors who are interested in computers, simulation, and/or Information Technology.
4. Institute of Defense Analysis (IDA), appealing to Applied Math and Quantitative Economics majors interested in Operations Analysis.
5. The Johns Hopkins University Applied Physics Laboratory (APL), for midshipmen in most technical majors interested in Operations Analysis and other technical applications.

A full listing of internships from all departments is available at http://www.usna.edu/AcResearch/MidshipmanInternships.html .

Circumcenters in 2, 3, and Higher Dimensions

Dr. Dale Peterson and Dr. Beth Schaubroeck
Department of Mathematical Sciences
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1. Introduction

A fundamental problem in astrodynamics is identifying an orbit from observations. Assuming a circular orbit, the problem is: Given three points, find the circle passing through them. We investigate how the problem can be solved from different perspectives: linear algebra, plane geometry, and calculus. We also investigate analogous results in higher dimensions. We identify some interesting symmetry and geometric interpretations.

These ideas are appropriate for study as special topics in courses such as multivariate calculus, introduction to proofs, or a majors' capstone project course. Motivated students could explore these topics, rigorously prove some of the methods that are sketched here, and extend the results.

2. The Circle

We are given three points in the plane that are not collinear, and wish to identify their *circumcircle* – the circle passing through them.

2.1 Linear System

An obvious method is to plug the three points into the equation $x^2 + y^2 + ax + by + c = 0$. You then have a linear system with three equations and three unknowns, *a*, *b*, and *c*. There is a unique solution (assuming, as we have, that the three points are not collinear).

Alternately, compute the following determinant and set it equal to zero. (Beginning linear algebra students are often taught this method as an application of determinants – see e.g. [1].)

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

This method is straightforward and mechanical, though it does not explicitly find the center or radius, and is geometrically unsatisfying.

2.2 Analytic Geometry and Calculus

In this approach we find the circle center; this with any one of the three points easily yields the circle equation. Given a line segment in the plane, its *perpendicular (line) bisector* is the line passing through the middle of the segment and perpendicular to it. We use a standard high school geometry result: Given any triangle, the perpendicular bisectors of the three sides intersect at a single point. This point is called the *circumcenter* of the triangle, and is the center of the circumcircle. (Note that this point need not be in the interior of the triangle.) A typical proof uses the congruence of the triangles created from the perpendicular bisector, the triangle side, and two of the triangle vertices (see Figure 1). By taking these two at a time, it follows that the distances from the triangle vertices to the intersecting point are the same, so they all intersect at the same point. [2]

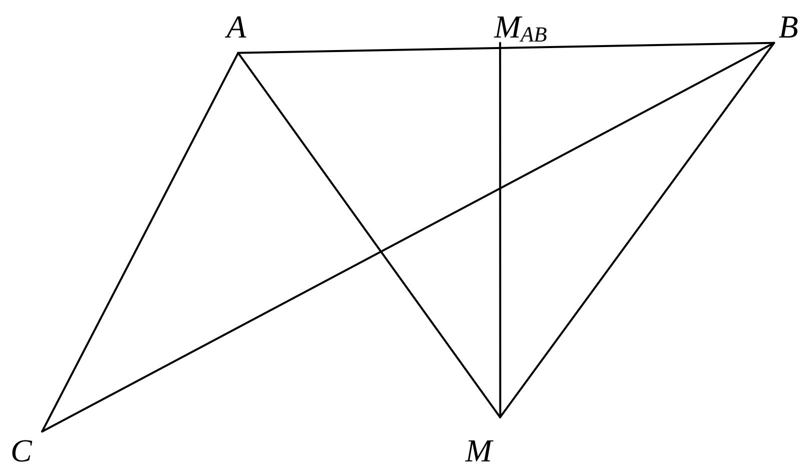

Fig. 1: Triangle $AM_{AB}M$ is congruent to triangle $BM_{AB}M$, so segments AM and BM are congruent.

Here is an alternative geometric proof. A perpendicular bisector is exactly the set of all points equidistant from the two endpoints. So, the center of any circle passing through the two points must lie on it. Thus any two perpendicular bisectors must intersect at the circumcircle center.

We sketch still another proof, based on calculus. Any side of a triangle is also a chord of the circumcircle. Any circle's chord is parallel to a tangent line of the circle, at the point where the chord's perpendicular bisector intersects the circle. (We leave the proof as an exercise to the reader; it may also be found in some calculus textbooks.) This perpendicular bisector is thus also perpendicular to the tangent line, and so is the same as the normal line to the tangent. Since a circle tangent's normal line passes through the circle center (a proof of which can be found in most any calculus text), any two normal lines intersect at the center.

Now we can find the intersection of any two of the three perpendicular bisectors – a 2×2 linear system (in contrast to the method of section 2.1, which is 3×3). If the three points are (x_1, y_1), (x_2, y_2), and (x_3, y_3), the midpoint of the segment joining (x_j, y_j) and (x_k, y_k) is $\left(\frac{x_j+x_k}{2}, \frac{y_j+y_k}{2}\right)$, and the slope of the perpendicular bisector is $\frac{x_j-x_k}{y_k-y_j}$. After a brief computation, we find that the equation of the perpendicular bisector is $y = \left(\frac{x_j-x_k}{y_k-y_j}\right)x - \frac{(x_j^2-x_k^2)}{2(y_k-y_j)} + \frac{y_k+y_j}{2}$. From here, a computer algebra system finds the intersection of any two of these lines. The coordinates of the circumcenter turn out to be:

x-coordinate: $\left(\frac{x_1^2(y_2-y_3)+x_2^2(y_3-y_1)+x_3^2(y_1-y_2)+(y_1-y_2)(y_1-y_3)(y_2-y_3)}{2(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))}\right)$

y-coordinate: $-\left(\frac{y_1^2(x_2-x_3)+y_2^2(x_3-x_1)+y_3^2(x_1-x_2)+(x_1-x_2)(x_1-x_3)(x-x_3)}{2(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))}\right)$

The symmetry here is surprising.

3. The Sphere

We are given four points in $\mathbf{R}^3$ that are not coplanar, and wish to identify their *circumsphere* – the sphere passing through them.

3.1 Linear System

Like section 2.1, plug the points into the equation $x^2 + y^2 + z^2 + ax + by + cz + d = 0$, yielding a 4×4 linear system. Alternately, compute the analogous 5×5 determinant and set it to zero.

3.2 Analytic Geometry and Calculus

We present two methods, both of which are extensions of section 2.2, but which alternately use lines and planes differently.

3.2.1 Intersecting Planes

Given a line segment in $\mathbf{R}^3$, its *perpendicular (plane) bisector* is the plane passing through the middle of the segment and perpendicular to it. Our four points define $\binom{4}{2} = 6$ line segments, each of which is a chord of the circumsphere. Now we use that the perpendicular plane bisector of any sphere chord passes through the sphere center. (We omit the proof.) Analogous to the 2-dimensional case, where we only needed two of the three perpendicular line bisectors to find the circumcircle center, here we do not need all six of the perpendicular plane bisectors to find the circumsphere center. We need three or four of the planes, depending on which line segments we use.

If we choose the three sides of a triangle, then their three perpendicular plane bisectors are linearly dependent and intersect at a line. This can be seen geometrically by projecting the plane triangle sides' perpendicular line bisectors out into a third dimension. These then become the perpendicular plane bisectors in $\mathbf{R}^3$, and the point where they intersect – the triangle's circumcenter – becomes the line where the three planes intersect. In this case, we need a fourth perpendicular plane bisector, from one of the other line segments.

If we choose three line segments that combined use all four points as endpoints, then their three perpendicular plane bisectors will intersect at a single point. (We leave the proof of this to the motivated student.) In any event, the perpendicular plane bisectors must come from line segments using all four points.

Like the 2-dimensional case, we can find the intersection of three linearly independent perpendicular plane bisectors. We start by considering the perpendicular plane bisector of the segment joining the points (x_j, y_j, z_j) and (x_k, y_k, z_k). It must go through the point $\left(\frac{x_j+x_k}{2}, \frac{y_j+y_k}{2}, \frac{z_j+z_k}{2}\right)$ and have normal vector $< x_k - x_j, y_k - y_j, z_k - z_j >$. Thus the equation of the plane is

$$(x_k - x_j)\left(x - \frac{x_j + x_k}{2}\right) + (y_k - y_j)\left(y - \frac{y_j + y_k}{2}\right) + (z_k - z_j)\left(z - \frac{z_j + z_k}{2}\right) = 0$$

or

$$(x_k - x_j)x + (y_k - y_j)y + (z_k - z_j)z - \tfrac{1}{2}\left({x_k}^2 - {x_j}^2 + {y_k}^2 - {y_j}^2 + {z_k}^2 - {z_j}^2\right) = 0.$$

Of course, we could have expected this form of the equation if we had rewritten the linear equation above as $(x_k - x_j)x + (y_k - y_j)y - \frac{1}{2}\left({x_k}^2 - {x_j}^2 + {y_k}^2 - {y_j}^2\right) = 0$. Again a computer algebra system solves the system of 3 equations to find the center of the circumsphere of the four points. A question for further investigation is whether the coordinates of the center of the circumsphere exhibit the same kind of symmetry as the coordinates of the center of the circumcircle in the planar case. *Mathematica* easily solves for the center of the circumsphere, but does not simplify the resulting answer very nicely.

Finally, this approach yields a 3-dimensional analog of the triangle circumcenter result: Given any four (non-coplanar) points, the six perpendicular plane bisectors of the points' six line segments all intersect at a common point – a *4-point circumcenter* – and this point is the center of the four points' circumsphere.

3.2.2 Intersecting Lines

In this approach, we choose three of the four points to define a plane, which cuts a cross-sectional circle in the circumsphere. The line through that circle's center – that is, through the three point's circumcenter – and perpendicular to the circle, which we call the circle's *normal line*, passes through the circumsphere center. We need any two of the $\binom{4}{3}$ = 4 planes to obtain two normal lines that will intersect at the circumsphere center. Once we have the two lines in parametric form, we can set, say, their *x*- and *y*-coordinates equal, and solve the 2×2 linear system to find the two parameters.

This method also has a calculus basis. Each cross-sectional circle of a sphere is parallel to the tangent plane to the sphere that is at the point where the circle's normal line intersects the sphere. Analogous to the 2-dimensional case, any two normal lines intersect at the sphere center.

This approach yields another 3-dimensional analog of the triangle circumcenter result. First we define the *normal line* of a triangle in $\mathbf{R}^3$ as the line through its circumcenter and perpendicular to the plane of the triangle. The result is then: Given any four (non-coplanar) points, the four normal lines of the points' four triangles all intersect at a common point, which is the same as the 4-point circumcenter found in section 3.2.1 above.

3.3 Relationship between the two methods

We note that the midpoint of a line segment can be thought of as the circumcenter of the two endpoints. This helps us identify a relationship between the two methods of section 3.2: We take the circumcenter of 2 and 3 points, respectively, and then take the normal to each – of dimension 2 and 1, respectively – through the circumcenter. This normal passes through the sphere center. We generalize this in section 4.2 below.

4. The *n*-dimensional Hypersphere

We are given (*n*+1) points in $\mathbf{R}^n$ that do not all reside in any (*n*-1)-dimensional hyperplane, and wish to identify the *circumhypersphere* passing through them.

4.1 Linear System

Like section 2.1 and 3.1, plug the points into the equation $x_1^2 + x_2^2 + \cdots + x_n^2 + a_1x_1 + a_2x_2 + \cdots + a_nx_n + b = 0$, yielding an (*n*+1)×(*n*+1) linear system. Alternately, compute the analogous (*n*+2)×(*n*+2) determinant and set it to zero.

4.2 Analytic Geometry and Calculus

By extending the approaches of sections 2.2 and 3.2, it appears that there are (*n*-1) different methods for finding the circumhypersphere center, one for each *i*, *i* = 2, 3, …, *n*. For each *i*, choose some combinations of *i* of the (*n*+1) points; there are $\binom{n+1}{i}$ such combinations. Each has a convex hull in a subspace of dimension (i-1), and a circumcenter, defined recursively (where $i \geq 3$) using circumcenters of (i-1) or fewer points, as was done in sections 3.2.1 and 3.2.2 for a 4-point circumcenter. Each has an (n-i+1)-dimensional *normal hyperplane* that is normal to it and through its circumcenter; that hyperplane passes through the circumhypersphere center. Completing our conjecture, (n-i+2) such normal hyperplanes – as long as they originate from convex hulls that together use all (n+1) points, and are linearly independent – intersect at a single point, which is the (*n+1*)-*point circumcenter* and the circumhypersphere center.

The case where $i = n$ corresponds to the calculus-based interpretation. Each n-point convex hull is parallel to a tangent hyperplane of the circumhypersphere, and the normal line through the circumcenter of the convex hull – which is the same as the normal line to the tangent hyperplane – passes through the hypersphere center.

5. Conclusion and Further Work

We have proposed methods for finding the center of a circumcircle, circumsphere, and circumhypersphere using analytic geometry. In the case of the circle, we obtained simplified expressions for the center coordinates, and found a nice symmetry in them. We have also found a 4-point / 3-dimensional analog to the triangle circumcenter result, and proposed its extension to any dimension.

Some possible future work, which we propose for motivated students as special topics in upper-level mathematics courses, includes:

- Explicitly list the methods for a 4-dimensional circumhypersphere.
- Determine if the symmetry of the center coordinates of a circumcircle extends to the center of a circumsphere, and in general to an n-dimensional circumhypersphere.
- Determine if there is a recursive expression for the coordinates of the center of a circumhypersphere, based on those of lower dimensions.
- Prove that the perpendicular plane bisector of any sphere chord passes through the sphere's center.
- Prove that, as stated in section 3.2.1, the perpendicular plane bisectors of any three line segments that combined use all four points as endpoints are linearly independent and intersect at a single point.
- Study linear independence of the normal hyperplanes for the different methods in n dimensions.
- Formalize the definition of an n-point circumcenter, based recursively on the circumcenters of combinations of fewer points, and explore more of its properties.
- Prove that our definition of an n-point circumcenter is well-defined, that is, it is the same point regardless of whether it is (recursively) based on 2-point, 3-point, …, or (n-1)-point circumcenters – and, that it is in fact the center of the n points' circumhypersphere.

- Explicitly outline and code algorithms for the methods for the circle and the sphere.
- Count operations of the algorithms and compare their efficiencies.

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EDITORIAL POLICY: (Revised in 2011)

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Although we issue calls for papers with a topical focus for each issue, *this practice does not preclude the publication of other information worth sharing*. Anything mathematical (proofs, problems, models, curriculum, history, biography, computing) remains in the purview of the bulletin, as it has since its inception in 1991.

While the missions of the mathematics departments at the service academies are quite similar, each has a different means of accomplishing its goals. By sharing information, we will be able to improve our programs by learning from each other. Hopefully, through Mathematica Militaris, these programs can continue to develop a common identity, gain recognition, and build an effective communication link.

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